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Multi-Period Matching

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Abstract

We examine a dynamic, multi-period, bilateral matching market, such as a labor market where workers are long-lived and production occurs over a period of time. We define and identify sufficient conditions for the existence of a dynamically stable matching. Our framework accommodates many forms of inter-temporal preference complementarities, including a taste for variety and a status-quo bias. Extensions of our model incorporating imperfect information and financial transfers are proposed. We relate our analysis to market unraveling and to common market design applications, including the medical residency match.

Keywords: Dynamic Matching, Two-sided Matching, Stability, Market Design, Uncertainty
JEL: C78, D47

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Gale and Shapley (1962) elegantly tackled the problems of "College Admissions and the Stability of Marriage." By privileging stability, their analysis suggests an immutability to a match's outcome. But, this is not what we often observe. Consider a few consequences of seemingly, or aspirationally, stable pairings:

1. After freshman year, a student transfers to another college.
2. After ten years of marriage, a couple divorces. Each then marries a new partner.
3. To repay her student loans, an MBA graduate works in management consulting for two years. Once debt free, she joins a start-up for a fraction of her old salary.

As illustrated by the preceding cases, three important characteristics color most economic and social relationships. First, relationships have a temporal component. They last multiple periods and they can be revised with the passage of time. Commitment is limited and intended long-term relationships—four years of college, a lifelong marriage, a committed career—often see interim revisions. Second, preferences are path dependent. Switching costs, a desire for variety, and inter-temporal financial constraints introduce chronological complementarities among outcomes. Finally, an agent is often uncertain about the future and refines his opinions as new information comes to light. Any analysis of a two-sided market where relationships are not ephemeral, as in the above examples, must address these features. A one-period model misses them all.

A central challenge in extending Gale and Shapley's (1962) static model beyond a single period is that there is no immediate analog to their stability condition in a dynamic, multi-period framework. As emphasized by Roth (2002) and others, the long-term viability of a market often hinges upon its ability to consistently coordinate upon a stable outcome. In this paper, we propose a new stability definition—*dynamic stability*—that confronts the richness of a multi-period economy. Addressing the above desiderata, it allows for limited commitment, accommodates history dependence, and builds upon a robust model of agent behavior and beliefs. Drawing on familiar intuitions, a market outcome is dynamically stable if at each moment in time it is individually rational and no pair of agents can arrange a mutually-preferable relationship plan conjecturing that the wider market evolves in an unfavorable manner. The notion is succinct and open to applications and extensions.

Although it draws on classic ideas, dynamic stability provides a level of generality absent from prior studies of multi-period matching economies. Many studies have defined stability within a particular multi-period application, such as daycare assignment (Kennes et al., 2014a), and thus are highly case specific. More abstract definitions, such as those proposed by Damiano and Lam (2005) or Kurino (2009), rely on strong assumptions concerning preferences or beliefs, hampering their application to many real-world problems. For instance, switching costs, status-quo bias, or a taste for variety are not compatible with the "time-separable" preference specification common to the literature (Pereyra, 2013; Kurino, 2014). The sufficient condition ensuring the existence of a dynamically stable matching, however, accommodates these features. Furthermore, dynamic stability can be easily employed in applications involving monetary transfers or preference uncertainty, features not examined previously in multi-period matching models. While our exposition here centers on a two-period setting, it is straightforward to extend our analysis to more than two periods, and we do so in Appendix B and in a companion paper (Kadam and Kotowski, 2015).[1]

Though we adopt Gale and Shapley's terminology of a matching between men and women, our model's applications extend beyond the study of interpersonal relations. Labor markets offer a germane application. According to the U.S. Bureau of Labor Statistics (2012), "individuals born from 1957 to 1964 held an average of 11.3 jobs from ages 18 to 46." Young workers, of course, move between jobs frequently. But, even middle-aged workers may have volatile employment arrangements, with 32.8 percent of jobs started by those aged 40 to 46 ending in less than a year (U.S. Bureau of Labor Statistics, 2012). Such dynamics have long been recognized by the labor-market search-and-matching literature (Rogerson et al., 2005), but they have been absent from studies of matching following Gale and Shapley (1962). Our model admits such career dynamics while incorporating common labor-market features, including learning-by-doing and non-constant wage profiles.

While our analysis is primarily a positive description of a multi-period economy, much research on matching markets is motivated by normative market design applications, such as the design of medical resident (Roth, 1984a) and student-school assignment schemes (Abdulkadiroğlu and Sönmez, 2003). Our model applies to these applications as well since they are multi-period problems. For example, a celebrated instance of market design is the National Resident Matching Program® (NRMP®) medical residency match (Roth and Peranson, 1999).[2] The NRMP is a clearinghouse that matches graduating medical students to hospital residency programs in the United States using an algorithm. While much attention is placed on the initial match of a trainee-doctor to a residency program, a deeper look at this market reveals a rich multi-period structure. This is hardly surprising as medical resi-

[1] Kadam and Kotowski (2015) investigate a special case of the model developed in this study with an emphasis on technical questions, such as the (non-)lattice structure of the set of dynamically stable matchings.

[2] "National Resident Matching Program" and "NRMP" are registered trademarks of the National Resident Matching Program.

dency is a long-term engagement. Some programs, for example, only provide introductory instruction (PGY-1[3]) and students must also match with a complementary advanced specialty (PGY-2). Others provide all years of training. Attrition and program switching occur as well (McAlister et al., 2008; Yaghoubian et al., 2012). To quote one resident's experience:

> S.M. matched to a preliminary year in Internal Medicine and an advanced position in Anesthesiology. Surprisingly, he found himself enjoying intern year much more than he had expected. At the end of intern year he moved on to Anesthesiology. Several months later S.M. realized he had been happier with the day-to-day work in Internal Medicine than in Anesthesiology. (. . .) [H]e arranged to finish the year in Anesthesiology and then return to the Internal Medicine program as a PGY-2. (Losada, 2010)

In light of such outcomes, an assessment of a match's success and of participants' welfare requires a longer-term perspective, which our model provides.[4] Its adaptation to other market design exercises, such as school assignment, follows accordingly.

Though our setting is too lean to address the details found in many market design applications, our analysis points to several themes deserving broader attention from scholars and practitioners. First, the multi-period nature of many situations introduces subtle complementarities in agents' preferences that may undermine an assignment's long-term stability. Mechanisms sensitive to this concern are therefore critical. Second, agents often participate in markets with limited knowledge about the future. When possible, proposed assignments should be robust to (or accommodate) agents' path dependent, post-match learning. Finally, as elaborated upon by Kadam and Kotowski (2015), (re-)matching frequency and assignment length are design variables. Novel solutions to otherwise complex problems may follow from their adjustment.

As suggested by the above examples, our study focuses on an economy where agents interact over multiple periods, forming and revising their relationships with time.[5] Similar two-sided,[6] one-to-one matching markets are studied by Damiano and Lam (2005) and

[3]"PGY-1" stands for Post Graduate Year 1. It is the first year of training doctors receive after graduating from medical school.

[4]Goldacre et al. (2010) conclude that about a quarter of doctors in the United Kingdom change specialty within the first ten years of their careers. We are unaware of analogous statistics for the United States.

[5]A complementary class dynamic models examines one-time matchings that arise over multiple periods. Here focus has ranged from questions of preference formation (Kadam, 2014) and unraveling (Roth and Xing, 1994) to managing the (stochastic) arrival and departure of agents or objects (Ünver, 2010; Leshno, 2012; Akbarpour et al., 2014; Doval, 2014; Thakral, 2015).

[6]One-sided markets, such as the house assignment problem, are a related class for analysis (Shapley and

Kurino (2009). More recently, several authors have considered multi-period, many-to-one or many-to-many matching models. Dur (2012), Bando (2012), Pereyra (2013), and Kennes et al. (2014a) propose models in this vein. Though these papers span a family of multi-period applications and environments, some even incorporating overlapping generations of agents, our model is not a special case of any of them. Notably, our model accommodates preferences featuring richer forms of inter-temporal complementarity, our stability concepts are distinct, and our algorithms for constructing stable assignments are new. We further contrast our analysis and the above studies in Section 2 after introducing our model. None of the preceding studies address monetary transfers or uncertainty and learning.

Though we stress the dynamic and sequential aspects of multi-period interactions, an interesting parallel exists between multi-period one-to-one matching markets and static many-to-many matching markets. Over a lifetime, each agent can have many partners. Recently, Hatfield and Kominers (2012b) have examined such markets in the matching with contracts framework (Hatfield and Milgrom, 2005). Dynamic matching problems can be analyzed within this paradigm by allowing agents to encode the date(s) of their relationship(s).[7] While we at times employ parallel reasoning, our model is not subsumed by their analysis.[8] The possible absence of a man-optimal stable matching (Example D.1) further contrasts our model with prior many-to-many matching models (Roth, 1984b).

This paper is organized as follows. In Section 1, we introduce our model. We define our preferred stability concept, dynamic stability, and we identify sufficient conditions for the existence of a dynamically stable matching. Our proofs are constructive, relying on new multi-period generalizations of Gale and Shapley's (1962) deferred acceptance algorithm. Importantly, the existence of a dynamically stable matching is not a consequence of a naive repetition of successive one-period matching markets. In fact, consecutive "spot markets" can generate unstable outcomes, kindle regret, and encourage strategic behavior. We illustrate these facts in Section 2 where we provide additional context for our analysis referring both to the existing literature and to common practice. With examples, we highlight the challenges of defining stability in a multi-period economy. We also explain how the NRMP matching algorithm, mentioned above, resolves the chronological complementarities among training programs and how it differs from our procedures. Sections 3 and 4 provide extensions to

Scarf, 1974; Hylland and Zeckhauser, 1979). Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2013), and Kurino (2014), among others, study dynamic variants of this problem. We do not study this case, though our analysis is complementary.

[7]Dimakopoulos and Heller (2014) adopt this approach to model the assignment of trainee lawyers to courts in Germany.

[8]Preferences fail substitutability, which is a key condition ensuring a stable matching's existence.

our model. In Section 3 we introduce incomplete information. We argue that re-matching after learning new information often fails to generate a Pareto improvement relative to an initial matching derived with imperfect information. We relate this observation to market unraveling, whereby agents commit to matching at earlier and earlier times. Finally, in Section 4 we consider monetary transfers, borrowing and savings. While expanding financial access and services is welfare enhancing in a partial-equilibrium sense, it can destabilize the market as a whole. Section 5 concludes. An online supplement collects omitted proofs (Appendix A), generalizes our model to T periods (Appendix B), analyzes our model's core (Appendix C), and presents additional examples and discussion (Appendices D and E).

1 The Model

Mindful of the noted applications, for expositional ease we present our model using Gale and Shapley's terminology of a matching between men and women. Synonyms for common applications would be students and schools, doctors and hospitals, or workers and firms. For brevity, we define some concepts only from the perspective of a typical man. Our model is symmetric and all definitions apply to women with obvious changes in notation. To streamline exposition, we postpone discussion of its subtleties to Section 2.

1.1 The One-Period Market

To introduce notation and to provide a benchmark, we briefly review Gale and Shapley's (1962) one-period matching market. There are finite, disjoint sets of men, $M = \{m_1, \ldots, m_{|M|}\}$, and women, $W = \{w_1, \ldots, w_{|W|}\}$. Each man (woman) can be matched to one woman (man) or not matched at all. By convention, a man (woman) who is not matched to a woman (man) is treated as matched to himself (herself). Thus, $W_m := W \cup \{m\}$ is the set of man m's potential partners, and w's potential partners are $M_w := M \cup \{w\}$. As there is only one period, each agent has a strict preference only over potential partners.

A matching is a function that assigns a partner to each agent. More formally, the function $\mu\colon M \cup W \to M \cup W$ is a *one-period matching* if $\mu(m) \in W_m$ for all $m \in M$, $\mu(w) \in M_w$ for all $w \in W$, and $\mu(i) = j \implies \mu(j) = i$ for all i. A stable matching cannot be blocked by any agent or pair. That is, (i) each agent weakly prefers his/her assigned partner to being not matched; and, (ii) no pair prefers to be together in lieu of their assigned partners.

Theorem 1 (Gale and Shapley (1962)). *There exists a stable one-period matching.*

To prove Theorem 1, Gale and Shapley introduce the (man-proposing) deferred acceptance algorithm, which proceeds as follows:

1. In round 1, each man proposes to his most preferred partner. Given the received proposals, each woman tentatively engages her most preferred suitor and rejects the others.

2. In round $\tau \geq 2$, each man proposes to his most preferred partner who has not yet rejected him. Each woman evaluates any received proposals and her engaged partner (if any) and tentatively engages her most preferred suitor and rejects the others.

The above process continues until no further rejections occur. At this point, engaged pairs are matched and all others remain unmatched. The resulting assignment is stable. As Roth (2008) explains, the algorithm has enjoyed wide application and we rely on it in many arguments to follow.[9]

1.2 A Multi-period Market

Extending the model, suppose agents interact over two periods. In every period, each man (woman) can be matched with one woman (man) or not matched at all. An agent's partners in periods t and t' may differ. We call this sequence of matchings a *partnership plan*. Thus, (j, k) is a partnership plan for i where he is matched with j in period 1 and with k in period 2. When confusion is unlikely, we write jk for (j, k). The plan jk is *persistent* if $j = k$. Else, it is *volatile*. Each agent has a strict and rational preference over partnership plans. If i prefers plan jk to plan $j'k'$, we write $jk \succ_i j'k'$. As usual, $jk \succsim_i j'k'$ if $jk \succ_i j'k'$ or $jk = j'k'$. The function $\mu \colon M \cup W \to (M \cup W)^2$ is a *multi-period matching* if for all i, $\mu(i) = (\mu_1(i), \mu_2(i))$ and μ_1 and μ_2 are one-period matchings. Henceforth, we refer to a multi-period matching simply as a *matching*.

Stability

In a one-period market, stability combines (i) an individual-rationality requirement and (ii) a pairwise no-blocking condition. The conditions assert that an agent or a pair cannot benefit by pursuing his/her/their best option outside of the market. A natural translation of these ideas to a multi-period setting begins from an ex ante perspective in period 1 and continues on a period-by-period basis. Adopting this approach, the matching μ is *ex ante individually*

[9] Appendix E describes the algorithm more formally and presents an example of its operation.

rational for agent i if he prefers $\mu(i)$ to his best unilateral outside option, which entails remaining unmatched. More formally, we say that agent i can *period-1 block* the matching μ if $ii \succ_i \mu(i)$. An individually rational matching cannot be period-1 blocked by any agent.

Similar logic guides blocking by a pair, though the set of available outside options is exponentially richer. If a pair blocks the matching μ in period 1, they pursue their most preferred arrangement among themselves in lieu of following the plan encoded in μ. They may aspire to a two-period partnership or they may decide upon a more unusual timing. Thus, the pair $(m, w) \in M \times W$ can *period-1 block* the matching μ if

1. $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$;
2. $wm \succ_m \mu(m)$ and $mw \succ_w \mu(w)$;
3. $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$; or,
4. $mm \succ_m \mu(m)$ and $ww \succ_w \mu(w)$.[10]

A matching is *ex ante stable* if it cannot be period-1 blocked by any agent or by any pair.

Though ex ante stability may be an appropriate solution concept for some applications, it presumes that agents can unequivocally commit to a matching, which may be rare in practice. Thus, our preferred stability definition, dynamic stability, eschews commitment and allows agents to block a matching conditional on the market's history. That is, agent i can *period-2 block* the matching μ if $(\mu_1(i), i) \succ_i \mu(i)$. Similarly, the pair (m, w) can *period-2 block* the matching μ if

1. $(\mu_1(m), w) \succ_m \mu(m)$ and $(\mu_1(w), m) \succ_w \mu(w)$; or,
2. $(\mu_1(m), m) \succ_m \mu(m)$ and $(\mu_1(w), w) \succ_w \mu(w)$.

A matching is *dynamically individually rational* if for all t it cannot be period-t blocked by any agent. A matching is *dynamically stable* if for all t it cannot be period-t blocked by any agent or pair. As explained in Section 2, dynamic stability balances behavioral plausibility, tractability, and extensibility and it differs from stability definitions encountered elsewhere in the literature on multi-period matching.

[10]Condition 4 may seem redundant. We include it for completeness since single-agent and pairwise definitions of blocking are special cases of more general coalition-based definitions (Appendix C).

Table 1: All ex ante stable matchings in Example 1.

Matching	m_1	m_2	w_1	w_2
μ^1	w_1w_1	w_2w_2	m_1m_1	m_2m_2
μ^2	w_1w_2	w_2w_1	m_1m_2	m_2m_1
μ^3	w_2w_1	w_1w_2	m_2m_1	m_1m_2

Example 1. Consider a market with two men and two women. Their preferences are:

$\succ_{m_1}: w_1w_1, w_1m_1, w_1w_2, m_1w_1, w_2w_1, m_1m_1$ $\quad\quad$ $\succ_{w_1}: m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1$

$\succ_{m_2}: w_2w_2, w_2m_2, w_2w_1, m_2w_2, w_1w_2, m_2m_2$ $\quad\quad$ $\succ_{w_2}: m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2$

Here, w_1w_1 is m_1's most preferred plan, w_1m_1 is second best, and so on. Unlisted plans are inferior to those listed and are not individually rational. This market has three ex ante stable matchings, which are listed in Table 1. To read the table, under μ^1, $\mu^1(m_1) = w_1w_1$, $\mu^1(m_2) = w_2w_2$, and so on. As clear from the table, both persistent and volatile matchings may be ex ante stable.

If we focus on the matching μ^2, we note that m_1 is paired with w_2 in period 2. However, $w_1m_1 \succ_{m_1} w_1w_2 = \mu^2(m_1)$. Thus, it seems unlikely m_1 would agree to a continuation of μ^2 once period 2 arrives. He prefers to renege on his commitment and the consequences for him of doing so are unambiguously positive. Thus, μ^2 is not dynamically stable. The remaining matchings, μ^1 and μ^3, are both dynamically stable. By allowing blocking in multiple periods conditional on the market's history, dynamic stability refines its ex ante counterpart through its accommodation of limited commitment. Intuitively, it offers the same type of refinement as provided by sub-game perfection to the Nash equilibria of an extensive-form game.

Existence of Stable Matchings

An important feature of both ex ante and dynamically stable matchings is that we can confirm their existence using generalizations of the deferred acceptance algorithm. As every dynamically stable matching is also an ex ante stable matching, we start by verifying the latter's existence. To do so, we employ the following algorithm, which will reappear below. In the plan deferred acceptance procedure (P-DA), each man proposes to one woman at a time and specifies their exclusive relationship's timing. A man and woman may be together for both periods or together in one period and single otherwise. Such proposals are made just like in the usual deferred acceptance algorithm with the woman tentatively accepting her best available option. Once no proposals are rejected, the final matching is set. Interestingly, our argument can be interpreted as men proposing from a restricted set of "contracts," as they may in the setting of Hatfield and Milgrom (2005). We then confirm that the resulting matching is stable when the restriction is removed.

Algorithm 1 (P-DA)**.** The *(man-proposing) plan deferred acceptance procedure* identifies a matching μ^* as follows. For each m let $X_m^0 = \cup_{w \in W}\{ww, wm, mw\}$. At $\tau = 0$, no plans in X_m^0 have been rejected. In round $\tau \geq 1$:

1. Let $X_m^\tau \subset X_m^0$ be the subset of plans that have not been rejected in some round $\tau' < \tau$. If $X_m^\tau = \emptyset$ or $mm \succ_m x$ for all $x \in X_m^\tau$, then m does not make any proposals. Otherwise, m proposes to the woman identified in his most preferred plan in X_m^τ. If ww is his most preferred plan, he proposes a two-period relationship to w. If wm (mw) is his most preferred plan, he proposes a one-period partnership with w for period 1 (2). In period 2 (1), both m and w are to be unmatched.

2. Let X_w^τ be the set of plans made available to w. If $ww \succ_w x$ for all $x \in X_w^\tau$, w rejects all proposals. Otherwise, w (tentatively) accepts her most preferred plan in X_w^τ and rejects the others. A woman may accept at most one plan at a time.

The above process continues until no rejections occur. If w accepts m's proposal in the final round, define $\mu^*(m)$ and $\mu^*(w)$ accordingly. If i does not make or receive any proposals in the final round, set $\mu^*(i) = ii$.

We will illustrate the P-DA's operation as part of Example 3 below. In the interim, we note that the procedure's outcome enjoys the following property.

Theorem 2. *The P-DA matching is ex ante stable.*

As with all formal results, we prove Theorem 2 in Appendix A. The argument proceeds along familiar lines. Any incipient blocking pair must meet in some round in the P-DA's operation, which implies that the woman must have rejected the man's proposal. As the woman's tentative matching only improves in later rounds of the P-DA, the woman must prefer the final outcome to the blocking arrangement.

The P-DA outcome is ex ante stable and may also be dynamically stable, but not always. At times, a dynamically stable matching may not exist.

Example 2. Consider a market with one man and one woman.[11] Their preferences are:

$$\succ_m: wm, ww, mm \qquad \succ_w: mm, ww$$

There are only two candidate stable matchings. The matching where $\mu(m) = mm$ is not ex ante stable as the couple can period-1 block it. The matching where $\mu'(m) = ww$ is ex ante stable. However, it is not dynamically stable since m will renege after period 1.

Example 2 suggests that agents' preferences must exhibit additional structure to ensure the existence of a dynamically stable matching. Ideally, this structure should be compatible with behavioral and economic characteristics common to dynamic markets, including a status-quo bias and complementarities among sequentially-assigned partners. The condition we propose, sequential improvement complementarity, allows these features, as we explain below.

Definition 1 (SIC)**.** The preference $\succ_i$ satisfies *sequential improvement complementarity* if

1. $jk \succ_i jj \succsim_i ii \implies kk \succ_i jj$;
2. $jk \succ_i ji \succ_i ii \implies kk \succ_i ji$; and,
3. $ik \succ_i ij \succ_i ii \implies kk \succ_i ij$.

Let $\mathcal{C}_i$ be the set of preferences satisfying SIC for agent i.

The economic content of SIC is concentrated in its first point.[12] This condition can be paraphrased as follows: If an agent prefers to switch assignments after period 1, rather than maintaining his initial assignment, then the change must be toward a "better" option. For instance, a worker might wish to move from an unpaid internship to a full-time job. The converse change, from a full-time job to an unpaid internship, is inferior to keeping the full-time position. Intuitively, the prospect of a better future assignment fuels the dynamic incentives supporting stable outcomes. Many common cases are compatible to with SIC:

1. $kk \succ_i jk \succ_i kj \succ_i jj$ — Agent i's preference is time separable and history independent. Irrespective of his period 1 match, he wants to match with k in period 2.

[11] Hatfield and Kominers (2012a) consider a similar example of a doctor and a hospital contracting morning and afternoon shifts. They suggest the doctor and the hospital should sign a unified contract covering both shifts. In our model, that suggestion corresponds to the ex ante stable outcome.

[12] SIC(2) and SIC(3) adapt SIC(1) for cases where the agent can be unmatched in one period.

2. $jk \succ_i kk \succ_i kj \succ_i jj$ — Agent i has a taste for variety since the volatile plan jk is most preferred.

3. $kk \succ_i kj \succ_i jj \succ_i jk$ — Agent i exhibits a status-quo bias (Samuelson and Zeckhauser, 1988). Conditional on matching with j in period 1, he wants to continue that relationship even though a match with k was superior ex ante. Given their empirical ubiquity, due to switching costs for example, we examine preferences with such "inertia" in greater detail below.

Theorem 3. *If agents' preferences satisfy SIC, there exists a dynamically stable matching.*

Theorem 3's proof is constructive using a generalized P-DA procedure. The plan deferred acceptance procedure with adjustment introduces an adjustment step among agents who are unmatched by the P-DA in period 2.

Algorithm 2 (P-DAA)**.** The *(two-period, man-proposing) plan deferred acceptance procedure with adjustment* identifies a matching μ^* as follows:

Step 1. Implement the P-DA procedure and call the resulting interim matching $\tilde{\mu}^1 = (\tilde{\mu}^1_1, \tilde{\mu}^1_2)$. For each agent i who is assigned a partner in period 2 ($\tilde{\mu}^1_2(i) \neq i$), set $\mu^*(i) = \tilde{\mu}^1(i)$ and exclude the agent from further consideration.

Step 2. For each remaining man, define a preference among the remaining women conditional on his interim assignment: $w \succ_m^{\tilde{\mu}^1_1(m)} w' \iff (\tilde{\mu}^1_1(m), w) \succ_m (\tilde{\mu}^1_1(m), w')$.[13] Define the remaining women's conditional preferences analogously. Next, implement Gale and Shapley's (1962) (man-proposing, one-period) deferred acceptance algorithm where each agent makes/accepts proposals according to his/her conditional preference, $\succ_i^{\tilde{\mu}^1_1(i)}$. If $\tilde{\mu}^2_2(\cdot)$ is the resulting one-period matching, for all agents involved set $\mu^*(i) = (\tilde{\mu}^1_1(i), \tilde{\mu}^2_2(i))$.

Remark 1. In Appendix B we present T-period generalizations of SIC, of the P-DAA procedure, and of Theorem 3. The intuition from the 2-period case carries over by induction.

The P-DAA operates in two steps, and both are required to ensure stability. Step 1 corresponds to the P-DA and secures the matching's ex ante stability. Step 2 improves upon the P-DA matching and ensures that the final outcome cannot be period-2 blocked.[14] The following example illustrates the importance of this second step. The economy's only dynamically stable matching features a cyclic assignment among two men and two women.

[13] Kennes et al. (2014a) present a similar definition when introducing their "isolated preference relation."

[14] All components of SIC are important. If exactly one component of SIC fails, a dynamically stable matching may not exist. See Example 2 and Examples D.3 and D.4 in Appendix D.

Example 3. There are three men and three women whose preferences satisfy SIC:

$$\begin{array}{ll}
\succ_{m_1}: w_1w_2, w_2w_2, w_1m_1, m_1m_1 & \succ_{w_1}: m_1m_1, m_1m_2, m_1w_1, w_1w_1 \\
\succ_{m_2}: w_2w_1, w_1w_1, w_2m_2, m_2m_2 & \succ_{w_2}: m_2m_2, m_2m_1, m_2w_2, w_2w_2 \\
\succ_{m_3}: w_1w_1, w_2w_2, w_3w_3, m_3m_3 & \succ_{w_3}: m_3m_3, w_3w_3
\end{array}$$

Table 2 summarizes the P-DAA's operation. The first step, coinciding with the P-DA, terminates in three rounds. In round 1, m_1 proposes a two-period partnership to w_2 and is rejected. The proposals of m_2 and m_3 are similarly rejected. By the third round, each man's proposal is accepted. The resulting interim matching is ex ante stable.

Since m_3 and w_3 are matched together in period 2, their final matching is set. The others move onto the P-DAA's second step. Their conditional preferences at $\tilde{\mu}_1^1(\cdot)$ are:

$$\succ_{m_1}^{w_1}: w_2, m_1 \qquad \succ_{m_2}^{w_2}: w_1, m_2 \qquad \succ_{w_1}^{m_1}: m_2, w_1 \qquad \succ_{w_2}^{m_2}: m_1, w_2$$

Conditional on $\tilde{\mu}_1^1(\cdot)$, m_1 and w_2 wish to match together for period 2. Similarly, m_2 and w_1 wish to match together. Of course, the deferred acceptance algorithm leads to this outcome. The final assignment is this economy's only dynamically stable matching.

Conditional on the period 1 match, the adjustment step in the P-DAA refines the assignments of agents who are unmatched in period 2. A natural further adjustment, which is not part of the P-DAA, is to attempt a similar improvement for agents who are unmatched in period 1 conditional on their period 2 assignment. By changing the interim status quo, this adjustment may unwittingly introduce instability.

Example 4. There is one man and three women whose preferences satisfy SIC.

$$\succ_{m_1}: \underset{\mu^2}{\boxed{w_1w_3}}, w_1w_2, \underset{\mu^1}{\underline{m_1w_2}}, w_2w_2, m_1m_1 \qquad \begin{array}{l}
\succ_{w_1}: \boxed{m_1w_1}, \underline{w_1w_1} \\
\succ_{w_2}: \underline{w_2m_1}, m_1m_1, \boxed{w_2w_2} \\
\succ_{w_3}: \boxed{w_3m_1}, m_1m_1, \underline{w_3w_3}
\end{array}$$

There are two dynamically stable matchings, marked in the preference list above. Neither matching Pareto dominates the other. The P-DAA identifies μ^1, which is underlined. It matches m_1 and w_2 only for period 2. This outcome suggests an obvious adjustment: m_1 and w_1 should partner for period 1 holding fixed their period 2 pairings. Following this adjustment, m_1's incentives for period 2 change and the resulting assignment is unstable.

Table 2: Operation of the P-DAA Procedure in Example 3.

Step	Round	Proposal Extended			Proposal(s) Received		
		m_1	m_2	m_3	w_1	w_2	w_3
1 (P-DA)	1	w_2w_2	w_1w_1	w_1w_1	$\{m_2m_2, m_3m_3\}$	m_1m_1	-
	2	w_1m_1	w_2m_2	w_2w_2	m_1w_1	$\{m_2w_2, m_3m_3\}$	-
	3	w_1m_1	w_2m_2	w_3w_3	m_1w_1	m_2w_2	m_3m_3
2 (Adjustment)	1	w_2	w_1	-	m_2	m_1	-
Final Matching		w_1w_2	w_2w_1	w_3w_3	m_1m_2	m_2m_1	m_3m_3

A Special Case: Rankings and Inertia

To highlight the properties of stable matchings, it is helpful to focus on a special case of our environment. This case straddles the time-invariant and inertia examples noted above and, therefore, provides a useful starting point for applied studies.

The simplest way to construct a multi-period preference begins with an agent's ranking of potential partners abstracting from all temporal considerations. We call such a single-period assessment by i a *spot ranking* and we denote it by P_i. If j is superior to k, we write jP_ik. Following custom, jR_ik if jP_ik or $j = k$. Given a spot ranking, it is natural that partnership plans with higher-ranked partners are preferred. We say that $\succ_i$ *reflects the spot ranking* P_i if jR_ij' and kR_ik', with at least one ranking being strict, imply that $jk \succ_i j'k'$.[15] Preferences that reflect a spot ranking may assume many forms. For example, the dictionary (lexicographic) preference,

$$\succ_i: jj, jk, jl, kj, kk, kl, lj, lk, ll, \ldots, \tag{1}$$

and its rhyming dictionary alternative,

$$\succ_i': jj, kj, lj, jk, kk, lk, jl, kl, ll, \ldots,$$

both reflect the spot ranking jP_ikP_il. Preferences in this class are often modeled with additively separable utility functions or with reference to history independence or time invariance.

Whereas preferences reflecting a spot ranking satisfy SIC, they preclude common cross-period complementarities. For example, workers often prefer longer-term employment with

[15] Reflection resembles Roth's (1985a) responsiveness. It differs since the timing of partners matters.

the same employer. Similarly, families often want their younger child (the period 2 match) to attend the same school as his older sibling (the period 1 match) (Dur, 2012). Such instances of status-quo bias or switching cost suggest that agents often favor persistent outcomes. A natural way to capture this idea is to allow persistent plans to (weakly) rise in rank relative to other plans. We say that the preference $\succ_i$ *exhibits inertia relative to* $\succ_i'$ if

- $jj \succ_i' kk' \implies jj \succ_i kk'$;
- $jj \succ_i' kk \iff jj \succ_i kk$; and,
- If $j \neq j'$ and $k \neq k'$, then $jj' \succ_i' kk' \iff jj' \succ_i kk'$.

For instance,

$$\succ_i'' : jj, jk, kk, jl, ll, kj, kl, lj, lk, \ldots$$

exhibits inertia relative to $\succ_i$, defined in (1), as kk and ll are relatively more preferred. Switching to j, the ex ante best match, after initial assignments to k or l is not desirable.

While any preference can be seasoned with extra inertia, we apply it to preferences that reflect a spot ranking. If $\Upsilon(\succ_i)$ is the set of preference profiles that exhibit inertia relative to $\succ_i$ and $\mathcal{S}_i$ is the set of preferences reflecting a spot ranking, then define $\bar{\mathcal{S}}_i := \cup_{\succ_i \in \mathcal{S}_i} \Upsilon(\succ_i)$ as the set of *preferences with inertia relative to* $\mathcal{S}_i$. Preferences in $\bar{\mathcal{S}}_i$ satisfy the "rankability" condition of Kennes et al. (2014a). In their model of daycare assignment, children's preferences satisfy this condition.[16] Unlike their definition, our construction decouples the ranking and the inertia elements of rankability, which facilitates this class' further generalization beyond two periods (Kadam and Kotowski, 2015).

Corollary 1. *$\mathcal{S}_i \subset \bar{\mathcal{S}}_i \subset \mathcal{C}_i$. Thus, if $\succ_i \in \bar{\mathcal{S}}_i$ for all i, a dynamically stable matching exists.*

While inertia biases preferences toward persistent plans, it hardly precludes volatile outcomes. For instance, in Example 1, $\succ_i \in \bar{\mathcal{S}}_i$ for all i yet both women prefer the volatile dynamically stable matching μ^3 to its persistent counterpart. Therefore, match volatility is not necessarily driven by an intrinsic preference for variety. Generally, it can occur despite a proclivity for persistence as a compromise among competing interests.

Beyond constituting a natural class of preferences for applied analysis, when $\succ_i \in \bar{\mathcal{S}}_i$ for all i the resulting economy enjoys several properties, some of which we highlight below.

[16] Kennes et al. (2014a) assume that priorities (daycares' "preferences") satisfy a more restrictive condition.

Simplifying the P-DAA Though the P-DAA procedure is an intuitively generalizes the deferred acceptance algorithm, its multi-step nature complicates its operation. When $\succ_i \in \bar{\mathcal{S}}_i$ for all i, the P-DAA enjoys a considerable simplification. First, elicit each agent's preference for persistent plans and define the *ex ante spot ranking induced by* $\succ_i$ as $jP_{\succ_i}k \iff jj \succ_i kk$.[17] Second, employ $P_{\succ_i}$ in the deferred acceptance algorithm to find a stable matching.

Algorithm 3 (E-DA)**.** The *(man-proposing) ex ante deferred acceptance procedure* assigns the one-period matching identified by the (man-proposing, one-period) deferred acceptance algorithm where each agent makes/accepts proposals according to $P_{\succ_i}$ in each period.

When $\succ_i \in \bar{\mathcal{S}}_i$ for all i, the E-DA and the P-DAA matchings coincide (Lemma A.3); hence, the E-DA outcome is dynamically stable. A laudable quality of the procedure is that it draws on relatively little information. Beyond $P_{\succ_i}$, an agent need not know or communicate $\succ_i$ in its entirety. In some applications this limited requirement may be advantageous.

Welfare Generally, a dynamically stable matching may not be Pareto optimal. In fact, it may be Pareto dominated by another dynamically stable matching (Kadam and Kotowski, 2015). The reason behind this conclusion is that dynamically stable matchings often feature collectively "cyclic" assignments, as in Examples 1 and 3. Cyclic assignments naturally arise in multi-period economies due to the coordination and scheduling aspects introduced by the time dimension. However, if preferences are restricted to $\bar{\mathcal{S}}_i$ we can conclude the following.

Theorem 4. *If $\succ_i \in \bar{\mathcal{S}}_i$ for all i, then every persistent dynamically stable matching is Pareto optimal.*

Noting the equivalence between the E-DA and the P-DAA procedures, a corollary to Theorem 4 is that the P-DAA matching is Pareto optimal when $\succ_i \in \bar{\mathcal{S}}_i$ for all i. More generally, the P-DAA matching may not be Pareto-optimal as the algorithm may pass-over a cyclic assignment that some agents may particularly enjoy (Example D.2). For brevity we do not pursue it here, but a further extension of the P-DAA can incorporate additional adjustment rounds to resolve such coordination failures. Necessarily, such adjustments become increasingly complex once we step beyond a 2-period setting.

Rescheduling and Timing Though key for welfare, scheduling is also central for market stability. Given a (two-period) matching, three schedule changes are possible: the initial

[17]Kennes et al. (2014a) present an analogous definition when introducing their "isolated preference relation." Of course, if $\succ_i$ reflects P_i, then $P_i = P_{\succ_i}$ (Lemma A.2).

matching can be prolonged, the final matching can be expedited, or matching times can be reversed. Of these, only the second is a dynamically stable proposition.

Theorem 5. *Suppose* $\succ_i \in \bar{\mathcal{S}}_i$ *for all* i *and let* $\mu = (\mu_1, \mu_2)$ *be a dynamically stable matching. Then* $\bar{\mu} = (\mu_2, \mu_2)$ *is also dynamically stable.*

An intuition for Theorem 5 can be found by recalling a finitely repeated non-cooperative game. In a Nash equilibrium of such a game, the players' final period actions must be a stage-game Nash equilibrium. Thus, repeating it in every period is also an equilibrium of the game as a whole. Similarly, when $\succ_i \in \bar{\mathcal{S}}_i$ for all i, repeating the final period matching is also stable. Despite preference inertia, the natural reciprocal to Theorem 5 is not generally true. Though qualified in Section 3, prolonging an assignment, (μ_1, μ_1), can beget instability. Likewise, reversing a stable matching's timing, (μ_2, μ_1), can be equally troublesome. Changing μ^3 in Example 1 in either manner leads to a dynamically unstable outcome.

2 Discussion

Having introduced our model, we would like to provide additional context for some of our definitions and conclusions. To emphasize key themes, we focus on two points. First, we contrast our matching procedures with commonly-proposed (and employed) alternatives. Thereafter, we explain how dynamic stability resolves the many trade-offs encountered when addressing "stability" in multi-period economies. On both matters there is much variance in the literature and in practice.

2.1 Multi-period Matching Mechanisms

A reassuring feature of our analysis is that a generalization of the deferred acceptance algorithm leads to a dynamically stable outcome. Thus, the P-DAA enjoys both a literal interpretation, as a centralized assignment mechanism, and a metaphorical one, as a model of a decentralized market where the proposing side enjoys market power. Of course, many multi-period generalizations of the deferred acceptance algorithm are possible. Below we sketch other common approaches, noting that they often fail to achieve a dynamically stable outcome. We also address the strategic issues that arise in multi-period setting.

Repeated, Single-Period Matching Markets

Likely the simplest multi-period generalization of the deferred acceptance procedure consists of a repetition of one-period matchings derived using Gale and Shapley's (1962) original algorithm. Intuitively, this approach captures the flavor of successive spot markets. At a high level, Damiano and Lam (2005), Kurino (2009), Dur (2012), Pereyra (2013), and Kennes et al. (2014a) rely on mechanisms implementing successive single-period stable matchings of this form. Successive assignments are derived conditional on past outcomes. Translating this idea to our setting requires care. When preferences are defined over partnership plans, there need not exist a stand-alone "single-period preference" that can be used to identify a "stable one-period matching." Like Kennes et al. (2014a), we consider the following operationalization.

Algorithm 4 (S-DA)**.** The *(man-proposing) spot-market deferred acceptance procedure* defines the matching $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$ on a period-by-period basis as follows:

Period 1. Define $\tilde{\mu}_1$ as the one-period matching identified by the (man-proposing) deferred acceptance algorithm where each agent i makes/accepts proposals according to his/her ex ante spot ranking, $P_{\succ_i}$.[18]

Period 2. Define agent i's *conditional spot ranking at* j as $kP^j_{\succ_i}l \iff (j,k) \succ_i (j,l)$. Set $\tilde{\mu}_2$ to be the one-period matching identified by the (man-proposing) deferred acceptance algorithm where each agent i makes/accepts proposals according to his/her conditional spot ranking at $\tilde{\mu}_1(i)$, $P^{\tilde{\mu}_1(i)}_{\succ_i}$.

When preferences reflect a spot ranking ($\succ_i \in \mathcal{S}_i$), then $P_{\succ_i} = P^j_{\succ_i}$. Thus, the S-DA matching coincides with the E-DA matching and is dynamically stable. However, once preferences exhibit inertia ($\succ_i \in \bar{\mathcal{S}}_i$), the S-DA matching may be unstable.

Example 5. Consider the following market where $\succ_i \in \bar{\mathcal{S}}_i$ for all i:

$$\begin{array}{ll} \succ_{m_1}: w_2w_2, w_1w_2, w_1w_1, \ldots & \succ_{w_1}: m_1m_1, m_2m_2, m_3m_3, m_1m_2, m_1m_3, \ldots \\ \succ_{m_2}: w_1w_1, w_3w_3, w_3w_1, \ldots & \succ_{w_2}: m_3m_3, m_1m_1, m_3m_1, \ldots \\ \succ_{m_3}: w_1w_1, w_2w_1, w_2w_2, \ldots & \succ_{w_3}: m_2m_2, \ldots \end{array}$$

[18]Recall that $jP_{\succ_i}k \iff jj \succ_i kk$. This parallels how a single-period preference can be inferred in an inter-temporal consumption problem. For example, if an agent's utility function is $U_i(x_1, x_2, \ldots) = \sum_t \delta^t u_i(x_t)$, we can identify his single-period preference $u_i(\cdot)$ by eliciting preferences over constant consumption streams.

As confirmed in Online Appendix E, the S-DA matching is

$$\tilde{\mu}(m_1) = w_1w_2 \qquad \tilde{\mu}(m_2) = w_3w_3 \qquad \tilde{\mu}(m_3) = w_2w_1$$
$$\tilde{\mu}(w_1) = m_1m_3 \qquad \tilde{\mu}(w_2) = m_3m_1 \qquad \tilde{\mu}(w_3) = m_2m_2$$

This matching is neither ex ante nor dynamically stable.[19] For example, m_1 and w_2 can period-1 block $\tilde{\mu}$. Instead, the dynamically stable E-DA matching is

$$\mu^*(m_1) = w_1w_1 \qquad \mu^*(m_2) = w_3w_3 \qquad \mu^*(m_3) = w_2w_2$$
$$\mu^*(w_1) = m_1m_1 \qquad \mu^*(w_2) = m_3m_3 \qquad \mu^*(w_3) = m_2m_2$$

This example also illustrates why the P-DAA procedure does not allow *all* agents to re-match conditional on their period-1 assignment. Here, that exercise's outcome coincides with the unstable S-DA matching.

Dynamic matching procedures that fail to account for complementarities between successive periods, like the S-DA, introduce an "exposure problem" into the matching process. The same issue arises in a multi-item auction where complementary goods are sold independently through multi-round procedures (Milgrom, 2000; Bulow et al., 2009). In Example 5, w_1 faces risk if she pursues a relationship with m_1, her favorite partner. In period 1 she is able to match with m_1, seemingly making progress toward her most preferred outcome, m_1m_1. Nevertheless, she faces risk concerning the durability of others' preferences. Others' changing opinions impose an externality on w_1, ultimately leading to disappointment. The P-DAA mitigates this problem by allowing some agents to match for period 2 from the outset.

Backward Induction and the NRMP

The P-DAA determines assignments chronologically. First, matchings for period 1, and possibly period 2, are specified. And then, period 2 adjustments are made. This operation contrasts with the backward induction reasoning common in multi-period scenarios. As a practical illustration of a mechanism using this latter approach, consider again the NRMP. Four program types participate in the NRMP's Main Residency Match® (Table 3).[20] Students can enter Categorical and Primary programs immediately after medical school (PGY-1)

[19] The S-DA procedure is a specialization of the DA-IP mechanism proposed by Kennes et al. (2014a) to assign children to daycares. To nest Example 5 in their framework, call men "children" and women "daycares" with unit capacity. All preferences satisfy their assumptions. If $\{P_{\succ_{w_1}}, P_{\succ_{w_2}}, P_{\succ_{w_3}}\}$ is the initial priority structure, the matching $\tilde{\mu}$ is "stable" in their sense of the term (Kennes et al., 2014a, Definition 8). Hence, our definitions of ex ante and dynamic stability are distinct from, and not weaker than, their proposal.

[20] "Main Residency Match" is a registered trademark of the National Resident Matching Program.

Table 3: Available and Filled Positions in the 2014 Main Residency Match.*

Program Type	Categorical / Primary	Preliminary†	Advanced	Physician	Total
Program Year	PGY-1	PGY-1	PGY-2	PGY-2	–
Starting Calendar Year	2014	2014	2015	2014	–
Positions Available	22,557	4,121	2,719	274	29,671
Positions Filled	22,149	3,538	2,592	211	28,490

* Adapted from National Resident Matching Program (2014a, Table 7).
† Sum of Medicine-Preliminary, OB/GYN-Preliminary, Pediatrics-Preliminary, Surgery-Preliminary, and Transitional programs.

and these programs lead to certification in their specialty, usually after 3–6 years. Preliminary programs provide one or two years of training, are open to students immediately after medical school (PGY-1), but do not lead to certification. Instead, they are prerequisites for Advanced programs, which students enter subsequently (PGY-2) for an additional 3–5 years of training. Physician positions are advanced positions (PGY-2) that start in the current year but are available only to students who have completed graduate medical education.[21] As noted in Table 3, non-Categorical positions constitute about 24 percent of available positions.

There is a clear sequential complementarity between Preliminary and Advanced training. The NRMP algorithm addresses this complementarity anti-chronologically. For each Advanced program in his rank order list, a student may also submit a supplemental ranking of Preliminary programs. (Submitting supplemental lists is not required.) If the algorithm matches an applicant to an Advanced program, it then attempts to match him to a Preliminary program from the associated supplemental ranking. Successfully matching to a Preliminary program is not assured and a student may be matched only to the Advanced program at the algorithm's conclusion.

This procedure may lead to outcomes reminiscent of S.M.'s experience recounted in the introduction. For example, consider a medical student applying to advanced programs a_2 and b_2 in cities a and b. Clearly, a Preliminary program in city a, call it a_1, complements the co-located Advanced program. Likewise, Preliminary program b_1 complements its co-located

[21] In their historical account of unraveling in the medical resident matching process, Roth and Xing (1994) note how some advanced specialties would match with students far in advance. "In [the matches for neurological surgery (PGY-2), otolaryngology (PGY-2 and PGY-3) and urology (PGY-3)], medical-school seniors obtain their second- and third-year employment from 18 to 30 months before they will begin work, and also *before* they will be matched to their PGY1 positions" (Roth and Xing, 1994, p. 1021, original emphasis). Though the institutional context now differs, such phenomena highlight this market's multi-period nature.

Advanced program in city b. For concreteness, suppose the student's (true) preference is

$$a_1a_2 \succ_i b_1b_2 \succ_i b_1a_2 \succ_i a_1b_2 \tag{2}$$

where the period 1 assignment corresponds to preliminary training and the period 2 assignment corresponds to advanced training. Within the NRMP, the student can submit supplemental rankings accompanying each advanced program. For example, a plausible submission given his preferences may be

$$a_2\{a_1 \hat{\succ}_i b_1\} \hat{\succ}_i b_2\{b_1 \hat{\succ}_i a_1\} \succ_i \cdots . \tag{3}$$

Since $a_1a_2 \succ_i b_1a_2$, $\{a_1 \hat{\succ}_i b_1\}$ is the supplemental list accompanying advanced program a_2. Suppose the NRMP algorithm successfully assigns the student to his most preferred advanced program, a_2, and then to his second-choice preliminary program, b_1. The b_1a_2 outcome creates a risk for dynamic instability. Like S.M. from the introduction, the student may seek to transfer at the preliminary program's conclusion since $b_1b_2 \succ_i b_1a_2$. If he succeeds in doing so, he is better off.[22] Of course, this revision may mask a broader welfare loss. The student's initial match to program a_2 may have displaced another student-doctor from that position, resigning her to a less-preferred program. A chain of further displacements may affect others as well.

It is important to stress, however, that the complementarity among Preliminary and Advanced programs is but one of many practical challenges that the NRMP matching process must address. Others include couples wishing to match together, programs requiring even or odd numbers of residents, and the transfer of unfilled positions among programs within the same hospital ("reversions"). In *theory* any of these features can compromise a matching's stability and welfare properties (Roth, 1996). In *practice*, however, the algorithm appears to successfully navigate the resulting portfolio of concerns.

Strategic Issues

Though we have discussed several multi-period matching procedures, we have abstracted from the strategic concerns arising in markets operating according to these principles. Our analysis presumes that each agent follows his true preferences, but it is reasonable to assume

[22]In the NRMP algorithm, students are the "proposers." Therefore, program b_2 may be willing to accommodate i in future years since it did not necessarily reject him during the initial NRMP match when (3) was the student's report. A waiver from the NRMP may be required to execute the change.

that an agent may wish to improve his outcome by mis-representing his interests. A matching mechanism or procedure is called *strategyproof* if it is a dominant strategy for each agent to truthfully reveal his preferences. Otherwise, some agent can manipulate the outcome.

In the one-period case, there does not exist a strategyproof matching mechanism that always yields a stable outcome (Roth, 1982). For example, in the man-proposing deferred acceptance algorithm it is a dominant strategy for each man to truthfully reveal his preferences. For the women, however, a worthwhile strategy often involves a "truncation" of preferences (Roth and Rothblum, 1999). This involves pretending that the least-desirable acceptable partners are unacceptable.

In a multi-period market, the scope for manipulation is considerably richer. For example, an agent receiving proposals may wish to claim her most preferred partner is not acceptable. Consider again the S-DA procedure and Example 5. Had w_1 shunned the period-1 proposal of m_1, her favorite partner, the S-DA would have matched her with m_2 in both periods which she prefers.[23] Curiously, in a dynamic market, a proposing agent may also wish to strategize. In the NRMP example above, if program b_2 considered the student acceptable, the student could have improved his match by claiming $b_2\{b_1 \mathrel{\hat{\succ}_i} a_1\} \mathrel{\hat{\succ}_i} a_2\{a_1 \mathrel{\hat{\succ}_i} b_1\}$ instead of (3).

More positive conclusions apply to the mechanisms we have proposed. Unqualified strategyproofness is impossible (Roth, 1982), but the P-DAA outcome can be implemented in a strategyproof manner for the proposing side when preferences satisfy our assumptions. If $\succ_i \in \bar{\mathcal{S}}_i$ for all i, implementing the E-DA suffices since the one-period deferred acceptance algorithm is strategyproof for the proposing side. More generally, we can conclude the following.

Theorem 6. *Suppose agents' preferences satisfy SIC. If agents can only communicate preferences that satisfy SIC, then the P-DAA is strategyproof for the proposing side.*

The restriction on announcements is necessary. Otherwise, every mechanism that identifies a dynamically stable matching (if one exists) can be manipulated by at least one man and by at least one woman (Example D.5).

2.2 Stability in Multi-period Matching Markets

While we consider dynamic stability to be the simplest and the most natural multi-period generalization of Gale and Shapley's original idea, it differs from existing proposals. As

[23] In a recent working paper, Kennes et al. (2014b) show that the scope for a beneficial strategic manipulation of their dynamic matching mechanism becomes small with increasing market size.

Damiano and Lam (2005) explain, defining stability in a multi-period economy requires resolving many degrees of freedom absent from the one-period case. Thus, many variations are possible.

Some differences among proposed stability concepts stem from application-specific features. In a school-choice application, for example, the stability/fairness of an assignment is typically ascertained on a period-by-period basis (Dur, 2012; Pereyra, 2013; Kennes et al., 2014a). Priorities are commonly defined in terms of single/next-period assignments rather than lifetime "enrollment plans" and, consequently, agents' incentives on each side of the market reference different time horizons. The resulting notions of blocking and stability differ from our's (see footnote 19 above). Beyond such model-specific elements, several general considerations emerge in multi-period economies. We address two of these below.

Pairwise vs. Multilateral Blocking

Following Gale and Shapley (1962), we build our stability concepts around pairwise blocking. In contrast, others have emphasized the importance of blocking actions by larger groups (Damiano and Lam, 2005; Kurino, 2009). This emphasis is motivated by the noted parallel between multi-period one-to-one matching markets and static many-to-many markets where such definitions are sometimes preferred (Roth, 1984b; Echenique and Oviedo, 2006; Konishi and Ünver, 2006).

For completeness, we provide coalition-based definitions of blocking in Appendix C where we study our economy's core.[24] Nevertheless, we prefer the pairwise specification for three reasons. First, stronger definitions may preclude meaningful applied analysis. Conditions ensuring the core's non-emptiness, for example, may not apply to an application of interest. Second, our definitions capture the practical nature of economic interaction in markets where large-scale coordination among agents is not possible, rarely observed, or illegal. Though a multi-period economy introduces some added scope for coordination, the practical difficulties commonly cited in the one-period case continue to apply. Finally, we wish to maintain a minimal departure from classic bilateral matching models, which motivate our analysis.

[24]Conditions ensuring the core's non-emptiness and coincidence with the set of dynamically stable matchings are stronger than SIC, but plausible in applications. They do not preclude complementarities among successive assignment. Volatile matchings may belong to the core.

Ambiguity vs. Perfect Foresight

In a multi-period economy, an agent needs to anticipate the future. For example, if he contemplates blocking a proposed matching, he must subscribe to some model concerning the economy's subsequent evolution. In principle, this counterfactual reasoning may lead to a complex chain of conjectures. Such forward-looking, higher-order matters are central to the stability concepts studied by Damiano and Lam (2005) and Kurino (2009), for example. Favoring spareness, dynamic stability models future developments through an implicit robustness criterion embedded in the definitions of blocking, whereby agents anticipate unfavorable future developments. The advantages of this approach, particularly when preferences are history dependent, are well-illustrated if we consider an agent blocking a proposed matching in period 1 only, as in the following example.

Example 6. Consider a market with two men and one woman. Their preferences are:

$$\begin{array}{l} \succ_{m_1}: m_1w_1, \underline{w_1w_1}, m_1m_1 \\ \succ_{m_2}: w_1w_1, m_2w_1, \underline{m_2m_2} \end{array} \qquad \succ_{w_1}: \underline{m_1m_1}, m_2m_2, w_1m_2, w_1m_1, w_1w_1$$

The plans defining this economy's only dynamically stable matching are underlined. Interestingly, $m_1w_1 \succ_{m_1} \underline{w_1w_1}$. This raises the question: *Should m_1 leave w_1 for period 1 only and then return?* Though a seemingly promising idea, and a frequently entertained possibility (Damiano and Lam, 2005; Bando, 2012; Kennes et al., 2014a), its reasonableness depends on the market's contemporaneous and subsequent development. Since preferences are history dependent, the possibilities and stories are many.

1. If w_1 is passive and is unmatched given m_1's period 1 absence, then the proposition is promising only if m_2 is equally unassertive and does not pursue a period 2 relationship with w_1. In this case, w_1 would accept m_1's return in period 2 since $w_1m_1 \succ_{w_1} w_1w_1$. If m_2 is alert and matches with w_1 in period 2, m_1 remains unmatched. In either case, w_1 feels regret as a matching with m_2 from the outset would have been preferable.

2. Anticipating regret, suppose w_1 reacts to m_1's period 1 absence from the market and pairs with m_2 in period 1 instead. Now, m_1 will lose-out in period 2. Conditional on her period-1 matching, w_1 wishes to maintain her partnership with m_2.

3. If there is uncertainty, matters are complicated further. For instance, if m_1 conjectures that $\succ_{w_1}: m_1m_1, m_2m_2, w_1m_2, w_1w_1, w_1m_1, \ldots$ he risks being rejected by w_1 in period 2 even if he believes m_2 is timid and resigned to remaining unmatched in both periods.

In practice, all of the preceding cases—and many more—are plausible models of this market's development.[25] A knotty selection problem follows. Dynamic stability posits that m_1 resolves this ambiguity conservatively. As implicit in the definition of period-1 blocking, m_1 assumes that his period 1 absence will precipitate unfavorable developments and he should not count on a subsequent chance to pair with w_1.

Generalizing the example's intuition, when an agent or a pair (or a coalition in Appendix C), block a proposed matching, our definitions assume that each agent believes the market will evolve in the most unfavorable manner (to them) in response to their deviation. In practice, of course, this implies exclusion from the wider market.[26] Excluded agents anticipate implementing the best continuation plan given their conjectured restricted circumstances.[27] The result is a robust, detail-free model rationalizing agents' behavior adaptable to many applications. It avoids elaborately prophetic counterfactual reasoning concerning the market's evolution or a coalition's credibility, which becomes taxing even in modestly-sized markets with short time horizons. Moreover, it ensures that dynamic stability can be applied when imperfect information further impedes such assessments. Case 3 in Example 6 above already hinted at this application and we elaborate upon it in the following section where we introduce incomplete information into our model.

3 Limited Information and Learning

A strength of our model is that it can easily incorporate further extensions. One such extension involves imperfect information and learning. Most agents enter into multi-period relationships with limited knowledge about future preferences. Marriages are announced and dissolved, employees change jobs, and students transfer schools as relevant facts emerge. Learning justifies interim relationship revisions, especially when they improve upon an initial assignment. Though we are sympathetic to this intuition, our analysis qualifies it considerably once a matching's overall stability is accounted for.

To appreciate some of the emergent implications, we amend our model as follows. Each

[25] And this is without acknowledging likely reputation concerns.

[26] In a market design application, the return of "non-cooperative" agents is controllable by the designer. Thus, market exclusion may be real rather than conjectured and is used in practice. For example, in the NRMP matching process, applicants/residency programs that fail to honor the prescribed outcome, without securing a waiver, may be barred from accepting alternative positions/applicants or participating in future NRMP Matches (National Resident Matching Program, 2014b).

[27] Similarly, in a repeated game players anticipate best responding to the planned punishment should they deviate from a proposed equilibrium.

agent i has a preference over partnership plans $\succ_i$, but does not know the complete ranking. Though initially partial, his knowledge will improve with time. Specifically, assume that at period 1 agent i knows the following:

(L1) His preferences have inertia, $\succ_i \in \bar{\mathcal{S}}_i$.

(L2) His ex ante spot ranking is $P_{\succ_i}$.

Given this limited information, there are many ex post preferences that the agent may actually hold. For example, if $jP_{\succ_i}kP_{\succ_i}l$ then the agent knows that $jj \succ_i kk \succ_i ll$, but the relative ranking of kj is unknown. As time passes, agent i learns more about his preferences.

(L3) If in period 1 agent i is assigned to k, he learns his preferences for plans of the form kl', for all l'.

Continuing the above illustration, after being matched with k, agent i could discover that

$$jj \succ_i kj \succ_i kk \succ_i ll \succ_i kl \succ_i \cdots .$$

Together, (L1)–(L3) outline a simple model of path-dependent learning. The situation is consistent with agent i learning about switching costs or the strength of preference inertia. In the above illustration, agent i knows that $jj \succ_i kk$, but is initially unsure whether switching to j in period 2 after being matched with k in period 1 is worthwhile. Given his period 1 knowledge, $kj \succ_i kk$ and $kk \succ_i kj$ are both plausible. He recognizes the true case only after a period-1 match to k. Beyond (L1)–(L3), we do not introduce further beliefs or priors.

To study a market where agents have limited information, we adopt a high-level perspective. Rather than introducing further ad hoc micro-level assumptions, we will model the market in reduced form by focusing on its final outcome. We assume that a market's outcome is the result of some matching mechanism. Though the term "matching mechanism" has the connotation of a centralized process, our intended meaning is broader. It should be interpreted as a black-box encompassing a pattern of regularized interaction leading to a matching in any economy. More formally, call the function $A(\cdot)$ a *matching mechanism* if it assigns a matching to each economy.[28] An economy is a tuple $e = (M, W, (\succ_i))$ encompassing sets of men and women along with their preferences. Thus, $A(e) = (A_1(e), A_2(e))$ is a matching among agents in e consistent with the interaction summarized by $A(\cdot)$. The P-DA, the P-DAA, the E-DA, and the S-DA are all examples of matching mechanisms.

[28] For simplicity, we do not consider random mechanisms.

Whereas matching mechanisms may differ along many dimensions, our restricted information structure draws attention to those with two pertinent properties. First, a reasonable mechanism should not leverage information that agents themselves do not know. In period 1, for example, agents know only their ex ante spot rankings. Thus, a matching mechanism should base its period 1 assignment only on that information and not on agents' ex post preferences. We call the matching mechanism *non-prophetic* if it assigns the same period 1 matching in all economies where agents' ex ante spot rankings coincide. Formally, matching mechanism $A(\cdot)$ is non-prophetic if for all economies $e = (M, W, (\succ_i))$ and $e' = (M, W, (\succ'_i))$ such that $P_{\succ_i} = P_{\succ'_i}$ for each i, $A_1(e) = A_1(e')$. The E-DA and the S-DA are both non-prophetic mechanisms.

Second, the mechanism should lead to a dynamically stable matching when $\succ_i \in \bar{\mathcal{S}}_i$ for all i, which is the assumed case. Dynamic stability remains a desirable benchmark in this setting since it captures an appealing no-regret property. To illustrate, observe that dynamic stability has both prospective and retrospective interpretations. In its traditional forward-looking form, an agent threatens to veto a matching that has not yet occurred. In its backward-looking form, dynamic stability captures how an agent feels ex post. While an agent cannot "turn back the clock" to period-1 block once on period 2's threshold, by (L3) he can assess a matching's continuation relative to persistent alternatives. For example, if after period 1 m discovers that in fact $ww \succ_m \mu(m)$ and w learns that $mm \succ_w \mu(w)$, both will feel regretful not pairing together at an earlier opportunity. A dynamically stable matching insulates agents from such regret. We call a matching mechanism *dynamically stable on* $\bar{\mathcal{S}}$ if it identifies a dynamically stable matching when $\succ_i \in \bar{\mathcal{S}}_i$ for all i. Many matching mechanisms are both non-prophetic and dynamically stable on $\bar{\mathcal{S}}$. The E-DA is an example, but there are others as well. Importantly, there exist such mechanisms that may result in volatile outcomes, perhaps by leveraging newly available information when setting the period 2 assignment.[29]

At this point, two natural questions arise. First, what are the properties of a stable matching in such market? And second, when is re-matching *after* period 2 preferences are known a welfare-enhancing proposition? We address these questions with two theorems.

[29]A simple example is the following: First, in the following economy

$$\succ_{m_1}: w_1w_1, w_1w_2, w_2w_2, w_2w_1, m_1m_1 \qquad \succ_{w_1}: m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1$$
$$\succ_{m_2}: w_2w_2, w_2w_1, w_1w_1, w_1w_2, m_2m_2 \qquad \succ_{w_2}: m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2$$

the mechanism assigns the dynamically stable matching where $\mu(m_1) = w_1w_2$ and $\mu(m_2) = w_2w_1$. In all other economies, it assigns the E-DA matching.

Theorem 7, which can be viewed as partial reciprocal to Theorem 5, begins with the stable matching $\mu = (\mu_1, \mu_2)$ generated by a non-prophetic mechanism. It shows that $\bar{\mu} = (\mu_1, \mu_1)$ is also dynamically stable. Thus, the theorem shows that in markets operating as we have assumed, forgoing a re-matching for period 2 is innocuous if stability is the sole concern. As shown by Example 3 above, this property is not generally true.

Theorem 7. *Let A be a non-prophetic matching mechanism that is dynamically stable on $\bar{\mathcal{S}}$. Suppose that in economy e where $\succ_i \in \bar{\mathcal{S}}_i$ for all i, $A(e) = \mu = (\mu_1, \mu_2)$. Then $\bar{\mu} = (\mu_1, \mu_1)$ is a dynamically stable matching in economy e.*

Since μ and $\bar{\mu}$ are both dynamically stable, they can be compared on an equal footing and we can tackle the second question posed above. While re-matching between periods offers an opportunity to improve welfare in light of newly available information, Theorem 8 shows that the scope for improvement is actually very small. In fact, re-matching cannot be Pareto improving relative to maintaining the interim status-quo. Furthermore, if an agent gains from re-matching, then his or her initial partner must necessarily be harmed by the ordeal—even when he or she finds a new period 2 partner.

Theorem 8. *Assume $\succ_i \in \bar{\mathcal{S}}_i$ for all i and suppose $\mu = (\mu_1, \mu_2)$ is dynamically stable. Let $\bar{\mu} = (\mu_1, \mu_1)$. If $\mu(i) \succ_i \bar{\mu}(i)$, then $\mu_2(i) = j$ and $\bar{\mu}(j) \succ_j \mu(j)$.*

An application unifying the preceding analysis considers market unraveling. Though unraveling may take on many forms, its most characteristic feature is the early commitment of parties to relationships far in the future, often *before* valuable information becomes known. It is known that uncertainty contributes to unraveling as early contracting provides insurance (Roth and Xing, 1994; Li and Rosen, 1998; Hałaburda, 2010; Ostrovsky and Schwarz, 2010; Echenique and Pereyra, 2013). Our model reinforces this intuition in a new way. Suppose agents interact on two occasions, in periods 1 and 2, and they learn new information before period 2. If a non-prophetic and dynamically stable (on $\bar{\mathcal{S}}$) mechanism describes this market's operation, agents will generally be averse to the prospect of revising their initial matching. Prolonging the initial period-1 matching is dynamically stable (Theorem 7); hence, they will not feel regretful ex post. Furthermore, half of agents who re-match between periods will be harmed by the change (Theorem 8). The high incidence of loss relative to the interim status-quo renders the existence and operation of a vibrant period-2 (re-)matching market quite precarious, despite the arrival of new information. Hence, any period-2 market has a natural inclination to thin out and to fold into the period-1 interaction.

The job market for entry-level lawyers in the United States approximates our two-period setting and illustrates the above phenomenon. This market de facto operates through the market for summer law interns in the preceding year (the period 1 matching).[30] Most firms have a summer program and extend job offers for the following year (the period 2 matching) to a high fraction (more than 90 percent) of summer interns (NALP, 2014). Thus, most firms and students elect to prolong their initial matching rather than waiting for additional information to arrive during the student's final year of law school.

4 Financial Transfers

An important generalization of Gale and Shapley's model involves financial transfers (Crawford and Knoer, 1981; Kelso and Crawford, 1982). Considering this extension in a multi-period setting introduces novel concerns not apparent in the one-shot, static case. For example, in a multi-period setting, agents' access to savings, credit, and other financial tools assumes practical prominence. The ability to shift income across time affects welfare directly through consumption smoothing. More subtly, however, this ability also has indirect consequences through its implications for incentives and, ultimately, market stability.[31] Our model lets us disentangle both direct and indirect effects.

For a unified exposition and notation, we continue to consider a market composed of men and women. Though intrahousehold transfers feature in many domestic arrangements, we hope this nomenclature does not shroud this extension's broader applicability. Instead, we may consider a matching between firms and workers, with transfers interpreted as wages, or a matching between students and schools, with transfers being tuition charges or scholarships.

Extending our original complete-information model, define agents' strict preferences over plans of the form

$$x = \big((i_1, y_1), (i_2, y_2)\big) \equiv \begin{pmatrix} i_1 & i_2 \\ y_1 & y_2 \end{pmatrix}.$$

A single-period assignment, (i_t, y_t), identifies an agent's period-t partner and a transfer received of a numeraire commodity in that period. An agent assigned the plan $\left(\begin{smallmatrix} j & k \\ 2 & -1 \end{smallmatrix}\right)$ is matched to j in period 1 and receives 2 units of the numeraire. In period 2, he is matched to k and supplies 1 unit of the numeraire. We continue to call $\mu_t \colon M \cup W \to M \cup W$ a single-period matching. We assume that period-t transfers between agents belong to the

[30]Roth and Xing (1994) and Ginsburg and Wolf (2004) describe this market in detail. Avery et al. (2001) examine the closely-related market for judicial law clerks.

[31]These conclusions echo Rogerson's (1985) from his analysis of a multi-period, principal-agent relationship.

finite set $Y \subset \mathbb{Z}$ and are specified by the function $\sigma_t \colon M \cup W \to Y$.[32] We assume that $0 \in Y$ and $y_t \in Y \iff -y_t \in Y$. The functions μ_t and σ_t are *compatible* if $\mu_t(i) = j \implies \sigma_t(i) = -\sigma_t(j)$. Thus, when together, a credit for i is a debit for j. When μ_t and σ_t are compatible, the pair $\rho_t = (\mu_t, \sigma_t)$ is a *single-period outcome*. An *outcome*, $\rho = (\rho_1, \rho_2) \equiv \left(\begin{smallmatrix} \mu_1 & \mu_2 \\ \sigma_1 & \sigma_2 \end{smallmatrix}\right)$, is a sequence of single-period outcomes.

While the above setup naturally generalizes our existing model, it remains incomplete. In multi-period economies, access to credit or savings affects behavior. To illustrate, suppose agent i's preference is $\left(\begin{smallmatrix} j & j \\ 2 & 2 \end{smallmatrix}\right) \succ_i \left(\begin{smallmatrix} k & k \\ 2 & 2 \end{smallmatrix}\right) \succ_i \left(\begin{smallmatrix} j & j \\ 3 & 1 \end{smallmatrix}\right)$. Absent further embellishments, if given a choice between $\left(\begin{smallmatrix} k & k \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} j & j \\ 3 & 1 \end{smallmatrix}\right)$, he should opt for the former. Suppose, however, that the agent can save part of his period-1 allocation of the transferable good. Now, if given a choice between $\left(\begin{smallmatrix} k & k \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} j & j \\ 3 & 1 \end{smallmatrix}\right)$, the latter is superior. Access to savings lets him independently transform the transfer stream $(3, 1)$ into $(2, 2)$ thereby replicating $\left(\begin{smallmatrix} j & j \\ 2 & 2 \end{smallmatrix}\right)$, which he prefers. From i's point of view, the plans $\left(\begin{smallmatrix} j & j \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} j & j \\ 3 & 1 \end{smallmatrix}\right)$ become "equivalent" once saving is possible.

While financial access can be subsumed into preferences a priori, we introduce it separately. Though this layering complicates some notation, it allows for welfare comparisons and new comparative statics with changing financial capabilities. As a first step, endow each agent with a *financial technology*, $f_i(\cdot) \colon Y \times Y \to 2^{Y \times Y}$. The set $f_i(y)$ consists of all independently attainable transfer profiles at $y = (y_1, y_2)$. Intuitively, $f_i(y)$ can be interpreted as agent i's intertemporal budget set. For example, if agent i can save without interest, we might define $f_i(y) = \{(y_1, y_2), (y_1 - 1, y_2 + 1), \ldots\}$. By saving, an agent can reduce the receipts in period 1 and increase them in period 2, as in the discussion above. Similarly, if the agent can borrow, we might define $f_i(y) = \{(y_1, y_2), (y_1 + 1, y_2 - 1), \ldots\}$. Other financial instruments can be modeled similarly. For simplicity, we assume that $y \in f_i(y)$ and we do not consider financial technologies with stochastic returns, such as stocks. Abusing notation, let $f_i\left(\left(\begin{smallmatrix} j & k \\ y_1 & y_2 \end{smallmatrix}\right)\right) \equiv \left\{\left(\begin{smallmatrix} j & k \\ y'_1 & y'_2 \end{smallmatrix}\right) \colon (y'_1, y'_2) \in f_i((y_1, y_2))\right\}$ be the set of attainable plans at $\left(\begin{smallmatrix} j & k \\ y_1 & y_2 \end{smallmatrix}\right)$.

To model decision making, define the f_i-*adaptation* of the preference $\succ_i$ as

$$\left(\begin{smallmatrix} j & k \\ y_1 & y_2 \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j' & k' \\ y'_1 & y'_2 \end{smallmatrix}\right) \iff \exists \left(\begin{smallmatrix} j & k \\ \tilde{y}_1 & \tilde{y}_2 \end{smallmatrix}\right) \in f_i\left(\left(\begin{smallmatrix} j & k \\ y_1 & y_2 \end{smallmatrix}\right)\right) \text{ such that}$$
$$\forall \left(\begin{smallmatrix} j' & k' \\ \tilde{y}'_1 & \tilde{y}'_2 \end{smallmatrix}\right) \in f_i\left(\left(\begin{smallmatrix} j' & k' \\ y'_1 & y'_2 \end{smallmatrix}\right)\right), \left(\begin{smallmatrix} j & k \\ \tilde{y}_1 & \tilde{y}_2 \end{smallmatrix}\right) \succ_i \left(\begin{smallmatrix} j' & k' \\ \tilde{y}'_1 & \tilde{y}'_2 \end{smallmatrix}\right).$$

If $x \succ_i^{f_i} x'$, then x gives agent i access to a better plan than x'. If $x \not\succ_i^{f_i} x'$ and $x' \not\succ_i^{f_i} x$, then x and x' are f_i-*equivalent* and we write $x \sim_i^{f_i} x'$.[33] This equivalence formalizes the

[32]The discrete set of feasible transfers may be denominated in the smallest practical unit, such as dollars or pennies. The integer restriction is without loss of generality.

[33]Lemma A.4 shows that $\sim_i^{f_i}$ is an equivalence relation. As usual, $x \succsim_i^{f_i} x'$ if $x \succ_i^{f_i} x'$ or $x \sim_i^{f_i} x'$.

observation from the motivating discussion above.

Despite the expanded domain, the definitions of blocking and stability retain their prior form with f_i-adapted preferences guiding behavior. Agent i can *period-1 block* ρ if $\left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right) \succ_i^{f_i} \rho(i)$ and he can *period-2 block* ρ if $\left(\begin{smallmatrix} \mu_1(i) & i \\ \sigma_1(i) & 0 \end{smallmatrix}\right) \succ_i^{f_i} \rho(i)$. Blocking by a pair generalizes similarly. The pair (m, w) can *period-1 block* ρ if there exist $y_1, y_2 \in Y$ such that

1. $\left(\begin{smallmatrix} m & m \\ y_1 & y_2 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} w & w \\ -y_1 & -y_2 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$;

2. $\left(\begin{smallmatrix} w & m \\ 0 & y_2 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} m & w \\ 0 & -y_2 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$;

3. $\left(\begin{smallmatrix} m & w \\ y_1 & 0 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} w & m \\ -y_1 & 0 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$; or,

4. $\left(\begin{smallmatrix} w & w \\ 0 & 0 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} m & m \\ 0 & 0 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$.

They can *period-2 block* ρ if there exists $y_2 \in Y$ such that

1. $\left(\begin{smallmatrix} \mu_1(w) & m \\ \sigma_1(w) & y_2 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} \mu_1(m) & w \\ \sigma_1(m) & -y_2 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$; or,

2. $\left(\begin{smallmatrix} \mu_1(w) & w \\ \sigma_1(w) & 0 \end{smallmatrix}\right) \succ_w^{f_w} \rho(w)$ and $\left(\begin{smallmatrix} \mu_1(m) & m \\ \sigma_1(m) & 0 \end{smallmatrix}\right) \succ_m^{f_m} \rho(m)$.

The outcome ρ is *ex ante stable* if it cannot be period-1 blocked by any agent or pair. It is *dynamically stable* if it cannot be period-t blocked by any agent or pair in any period.

Like in our original model, ex ante stable outcomes exist without further qualifications (Lemma A.5), but a restriction is required to ensure the existence of a dynamically stable outcome. However, as illustrated by the next example, transfers and financial access introduce additional complications due to their impact on dynamic incentives.

Example 7. Consider an economy with one employer (m) and one worker (w) with preferences given by (4) and (5), respectively.

$$\cdots \succ_m \left(\begin{smallmatrix} w & w \\ -1 & -2 \end{smallmatrix}\right) \succ_m \left(\begin{smallmatrix} w & w \\ -2 & -2 \end{smallmatrix}\right) \succ_m \left(\begin{smallmatrix} m & m \\ 0 & 0 \end{smallmatrix}\right) \succ_m \cdots \succ_m \left(\begin{smallmatrix} w & w \\ -1 & -3 \end{smallmatrix}\right) \succ_m \left(\begin{smallmatrix} w & w \\ -3 & -1 \end{smallmatrix}\right) \succ_m \cdots \quad (4)$$

$$\cdots \succ_w \left(\begin{smallmatrix} m & w \\ 1 & 1 \end{smallmatrix}\right) \succ_w \left(\begin{smallmatrix} m & m \\ 2 & 2 \end{smallmatrix}\right) \succ_w \left(\begin{smallmatrix} m & m \\ 3 & 1 \end{smallmatrix}\right) \succ_w \left(\begin{smallmatrix} m & m \\ 1 & 3 \end{smallmatrix}\right) \succ_w \left(\begin{smallmatrix} w & w \\ 0 & 0 \end{smallmatrix}\right) \succ_w \cdots \succ_w \left(\begin{smallmatrix} m & w \\ 2 & 0 \end{smallmatrix}\right) \succ_w \cdots \quad (5)$$

In this situation, the employer must hire the worker for two periods to produce output, but is unable to pay more than 2 dollars in wages per period. To work in both periods, the worker demands a lifetime income of 4 dollars. However, her most preferred outcome is to work for one period and then retire, provided she gets some income in retirement. Consider the following three possibilities:

1. Suppose neither party can save or borrow. In this case there is a unique dynamically stable outcome. The worker is employed for both periods at a wage of 2 per period: $\rho^*(m) = \left(\begin{smallmatrix} w & w \\ -2 & -2 \end{smallmatrix}\right)$ and $\rho^*(w) = \left(\begin{smallmatrix} m & m \\ 2 & 2 \end{smallmatrix}\right)$.

2. Suppose w gains access to savings, $f_w(y) = \{(y_1, y_2), (y_1 - 1, y_1 + 1), \ldots\}$, but m does not. Thus, $\left(\begin{smallmatrix} m & w \\ 1 & 1 \end{smallmatrix}\right) \sim_w^{f_w} \left(\begin{smallmatrix} m & w \\ 2 & 0 \end{smallmatrix}\right) \succ_w^{f_w} \left(\begin{smallmatrix} m & m \\ 2 & 2 \end{smallmatrix}\right)$ and w will period-2 block any plan where she receives 2 in period 1. Thus, there is no dynamically stable outcome.

3. Suppose, additionally, that $f_m(y) = \{(y_1, y_2), (y_1 - 1, y_1 + 1), \ldots\}$ and m can delay payments to w. Therefore, $\left(\begin{smallmatrix} w & w \\ -1 & -3 \end{smallmatrix}\right) \sim_m^{f_m} \left(\begin{smallmatrix} w & w \\ -2 & -2 \end{smallmatrix}\right)$. Now the outcome where $\rho^*(m) = \left(\begin{smallmatrix} w & w \\ -1 & -3 \end{smallmatrix}\right)$ and $\rho^*(w) = \left(\begin{smallmatrix} m & m \\ 1 & 3 \end{smallmatrix}\right)$ is dynamically stable. The employer prevents the worker from quitting prematurely by backloading her compensation.

As apparent in Example 7, transfers and finance have cross-cutting implications for market stability and welfare. Generally, credit or savings should improve welfare as they expand an agent's consumption possibilities. However, once we focus on stable outcomes, matters may be different. Stable outcomes may fail to exist. And, even if they do exist, welfare may decline. The dynamically stable outcome in the absence of savings (case 1) Pareto-dominates the outcome when saving is possible (case 3).

To ensure the existence of a dynamically stable outcome, and to resolve the complications highlighted by Example 7, we can mirror our original analysis. First, we generalize SIC.

Definition 2 (G-SIC)**.** The f_i-adapted preference $\succ_i^{f_i}$ satisfies *generalized sequential improvement complementarity* if

1. $\left(\begin{smallmatrix} j & k \\ y & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j & j \\ y & y' \end{smallmatrix}\right) \succsim_i^{f_i} \left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} k & k \\ \tilde{y}' & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j & j \\ y & y' \end{smallmatrix}\right)$;
2. $\left(\begin{smallmatrix} j & k \\ y & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j & i \\ y & 0 \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} k & k \\ \tilde{y}' & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j & i \\ y & 0 \end{smallmatrix}\right)$; and,
3. $\left(\begin{smallmatrix} i & k \\ 0 & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} i & j \\ 0 & y' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} k & k \\ \tilde{y}' & \tilde{y}' \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} i & j \\ 0 & y' \end{smallmatrix}\right)$.

G-SIC and SIC share a common underlying intuition and the former reduces to the latter when transfers are not possible ($Y = \{0\}$). Unlike SIC, however, G-SIC is a joint restriction on both preferences ($\succ_i$) and the financial technology (f_i). When f_i-adapted preferences satisfy G-SIC, the existence of a dynamically stable outcome follows from a generalized P-DAA procedure. We define this procedure in Appendix A and its operation parallels that of the original P-DAA with transfers incorporated into proposals (Crawford and Knoer, 1981; Kelso and Crawford, 1982). Verifying stability mimics the proof of Theorem 3.

Theorem 9. *If agents' preferences satisfy G-SIC, a dynamically stable outcome exists.*

An alternative route to ensure the existence of a dynamically stable matching is to specify collective preference/financial restrictions. Preferring to focus on agent-level assumptions, we do not pursue this reasoning here, though a reexamination of cases two and three in Example 7 points toward the argument involved. Once m gains access to savings, he can provide w adequate incentives given her financial technology. A collective restriction on $\{\succ_i^{f_i}\}$ ensuring such flexibility would again ensure existence. Backloaded compensation, end of contract bonuses, and vesting periods reflect this type of solution in practice.

5 Concluding Remarks

We have proposed a conservative, portable, multi-period generalization of the classic model of one-to-one matching. Our analysis shifts focus away from a one-shot interaction toward a long-term appraisal of a market's operation and of agents' welfare. Such a focus is the natural one since many bilateral interactions, such as marriage, employment, or schooling, are far from fleeting, though often impermanent. The conditions supporting stable outcomes are behaviorally-plausible and, we contend, quite common. Sequential improvement complementarity allows for complementarities among distinct partners and status-quo bias. While common attitudes, such as those featuring inertia, seemingly tilt preferences toward persistent plans, preferred stable outcomes may in fact be volatile. Though our discussion focuses on two periods, these conclusions generalize. Our model readily accommodates monetary transfers, credit and finance, and the common case where agents are uncertain about their future preferences. These extensions provide subtle qualifications of our primary analysis.

For brevity we have suppressed many natural embellishments. For example, we have not addressed the arrival or departure of agents nor the many-to-one nature of some matching problems. While such extensions introduce novel concerns, the motivations and essence of our stability definitions carry over naturally. Similarly, we have only sketched the strategic nuances of multi-period markets. We consider these questions as promising areas for future research and we hope that our analysis provides a foundation for their investigation. Of course, many of these features will also be central when the time-component of relationships is incorporated into market design exercises. In that context, time can serve as a design variable rather than a constraint. New solutions to previously challenging problems may emerge.

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The following appendices are intended for online publication only.

A Proofs and Omitted Lemmas

Proof of Theorem 2. Let μ^* be the matching identified by the P-DA procedure. According to the procedure, no agent will be assigned to a plan that is worse than being unmatched in both periods. Thus, $ii \not\succ_i \mu^*(i)$. Also, no pair $\{m, w\}$ can block μ^*. If not, then there exists a plan $x \in \{wm, mw, ww\}$ for m and a corresponding plan for w that both prefer. If $x \succ_m \mu^*(m)$, then m must have proposed that arrangement to w at some round before he made his proposal defined in $\mu^*(m)$. w must have rejected that original proposal; thus, she must prefer $\mu^*(w)$. However, this is a contradiction. $\square$

Proof of Theorem 3. Let μ^* be the matching identified by the P-DAA procedure. Let $\tilde{\mu}^1$ be the interim matching identified by the P-DAA's first step. As a preliminary observation, note that $\mu^*(i) \succsim_i \tilde{\mu}^1(i)$ for all i.[34]

Hence, μ^* cannot be period-1 blocked. If this was not the case, then the same blocking agent/pair can period-1 block $\tilde{\mu}^1$, which is not possible. (The P-DA procedure identifies an ex ante stable matching.)

Suppose agent i can period-2 block μ^*, i.e. $(\mu_1^*(i), i) \succ_i \mu^*(i)$. If $\mu_1^*(i) = i$, then $ii \succ_i \mu^*(i)$, which is a contradiction. Therefore, $\mu_1^*(i) = j \neq i$. There are two cases. If $\tilde{\mu}^1(i) = jj$, then by SIC(1) $ji \succ_i \mu^*(i) \succsim_i \tilde{\mu}^1(i) = jj \implies ii \succ_i jj = \tilde{\mu}^1(i)$, which is a contradiction. If instead $\tilde{\mu}^1(i) = ji$, then $ji \succ_i \mu^*(i) \succsim_i \tilde{\mu}^1(i) = ji$, which is also contradiction. Therefore, agent i cannot period-2 block μ^*.

If m and w can period-2 block μ^*, then $(\mu_1^*(m), w) \succ_m \mu^*(m) \succsim_m \tilde{\mu}^1(m) \succsim_m mm$ and $(\mu_1^*(w), m) \succ_w \mu^*(w) \succsim_w \tilde{\mu}^1(w) \succsim_w ww$. If $\mu_1^*(m) = w$ and $\mu_1^*(w) = m$, then m and w can period-1 block $\tilde{\mu}^1$, which is not possible. The same applies when $\mu_1^*(m) = m$ and $\mu_1^*(w) = w$. Thus, without loss of generality, there are two cases.

First, suppose $\mu_1^*(m) = m$ and $\mu_1^*(w) = m' \neq m$. In this case, $mw \succ_m \mu^*(m) \succsim_m mj = \tilde{\mu}^1(m) \succsim_m mm$ and $m'm \succ_w \mu^*(w) \succsim_w m'k = \tilde{\mu}^1(w) \succ_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = m$, SIC(1) implies that $ww \succ_m mm = \tilde{\mu}^1(m)$. If $j = w' \neq m$, SIC(3) implies

[34] If $\mu^*(i) = (\tilde{\mu}_1^1(i), \tilde{\mu}_2^2(i)) \neq \tilde{\mu}^1(i)$, then it must be that $\tilde{\mu}_2^1(i) = i$. Since the Gale and Shapley (1962) deferred acceptance procedure generates an individually rational (one-period) matching, $\tilde{\mu}_2^2(i) \succsim_i^{\tilde{\mu}_1^1(i)} i \implies \mu^*(i) = (\tilde{\mu}_1^1(i), \tilde{\mu}_2^2(i)) \succsim_i (\tilde{\mu}_1^1(i), i) = \tilde{\mu}^1(i)$.

$ww \succ_m mw' = \tilde{\mu}^1(m)$. Similarly, if $k = w$, then by SIC(2) $m'm \succ_w m'w \succ_w ww \implies mm \succ_w m'w = \tilde{\mu}^1(w)$. Finally, if $k = m'$, SIC(1) implies $mm \succ_w m'm' = \tilde{\mu}^1(w)$. Whatever the case, m and w can period-1 block $\tilde{\mu}^1$—a contradiction.

Instead, and second, suppose $\mu_1^*(m) = w' \neq w$ and $\mu_1^*(w) = m' \neq m$. In this case, $w'w \succ_m \mu^*(m) \succsim_m w'j = \tilde{\mu}^1(m) \succ_m mm$ and $m'm \succ_w \mu^*(w) \succsim_w m'k = \tilde{\mu}^1(w) \succ_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = m$, SIC(2) implies that $ww \succ_m w'm = \tilde{\mu}^1(m)$. If $j = w' \neq m$, SIC(1) implies that $ww \succ_m w'w' = \tilde{\mu}^1(m)$. By the same reasoning, we conclude that $mm \succ_w m'k = \tilde{\mu}^1(w)$ for $k \in \{w, m'\}$. Thus, m and w can period-1 block $\tilde{\mu}^1$, which is a contradiction. □

Lemma A.1. *Suppose* $\succ_i \in \bar{\mathcal{S}}_i$.

1. $jj \succ_i kk \implies jj \succ_i jk$ *and* $jj \succ_i kk \implies jj \succ_i kj$.
2. $jk \succ_i kk \implies jj \succ_i kk$ *and* $kj \succ_i kk \implies jj \succ_i kk$.
3. *For all* $l \neq j$ *and* $l \neq k$, $jj \succ_i kk \iff lj \succ_i lk$ *and* $jj \succ_i kk \iff kl \succ_i kl$.

Proof. Let $\succ_i' \in \mathcal{S}_i$ be such that $\succ_i \in \Upsilon(\succ_i')$.

1. $jj \succ_i kk \iff jj \succ_i' kk \implies jP_{\succ_i'}k \implies jj \succ_i' jk \implies jj \succ_i jk$. The second implication follows similarly.
2. $jk \succ_i kk \implies jk \succ_i' kk \implies jP_{\succ_i'}k \implies jj \succ_i' kk \implies jj \succ_i kk$. The second implication follows similarly.
3. $jj \succ_i kk \iff jj \succ_i' kk \iff jP_{\succ_i}k \iff jP_{\succ_i'}k \implies lj \succ_i' lk \implies lj \succ_i lk$ since $l \neq j, k$. The second implication follows similarly.

□

Proof of Corollary 1. By Lemma A.1, if $\succ_i \in \bar{\mathcal{S}}_i$, then either $jj \succ_i jk$ or $kk \succ_i jk$. Hence, $\bar{\mathcal{S}}_i \subset \mathcal{C}_i$. □

Lemma A.2. *If* $\succ_i$ *reflects* P_i, *then* $P_i = P_{\succ_i}$.

Proof. By definition, $P_i = P_{\succ_i}$ if and only if $jP_ik \iff jP_{\succ_i}k$. Suppose $\succ_i$ reflects P_i. Given $\succ_i$, $P_{\succ_i}$ is uniquely defined. Without loss of generality assume $jP_{\succ_i}k$. Suppose, for contradiction, that kP_ij. Then, $kP_ij \implies kk \succ_i jj \implies kP_{\succ_i}j \implies \neg[jP_{\succ}k]$. The first implication is from the definition of P_i. The second follows from the definition of $P_{\succ_i}$. The final implication, a contradiction, is because $P_{\succ_i}$ is asymmetric. Therefore, $jP_{\succ_i}k \implies jP_ik$. The converse case follows similarly. Therefore, $P_i = P_{\succ_i}$. □

Proof of Theorem 4. Let μ^* be a persistent dynamically stable matching. Suppose μ^* is not Pareto optimal. If it is Pareto-dominated by a persistent matching $\tilde{\mu}$, then there must exist $m \in M$ and $w \in W$ such that $\tilde{\mu}(m) = ww \succ_m \mu^*(m)$ and $\tilde{\mu}(w) = mm \succ_w \mu^*(w)$. However, this contradicts μ^* being dynamically stable since m and w can period-1 block μ^*.

Thus, μ^* must be Pareto-dominated by a matching $\tilde{\mu}$ which is volatile for some man, m_1. This implies $\tilde{\mu}(m_1) = m_1w_1$ or w_1m or w_1w_2. In the first two cases, $\tilde{\mu}(m_1) \succ_{m_1} \mu^*(m_1) \succsim_{m_1} m_1m_1$ and $\succ_{m_1} \in \bar{\mathcal{S}}_{m_1}$ imply that $w_1w_1 \succ_{m_1} \tilde{\mu}(m_1)$. In the third case, since $\succ_{m_1} \in \bar{\mathcal{S}}_{m_1}$, (i) $w_1w_1 \succ_{m_1} w_1w_2$ or (ii) $w_2w_2 \succ_{m_1} w_1w_2$. In all cases, therefore, we observe that $\tilde{\mu}(m_1)$ is dominated by a persistent plan. Without loss of generality, suppose $w_1w_1 \succ_{m_1} \tilde{\mu}(m_1) \succ_{m_1} \mu^*(m_1)$. As $\succ_{w_1} \in \bar{\mathcal{S}}_{w_1}$,

$$m_2m_2 \succ_{w_1} m_1m_2 = \tilde{\mu}(w_1) \succ_{w_1} \mu^*(w_1) \succ_{w_1} m_1m_1$$

for some $m_2 \in M$, $m_2 \neq m_1$. Otherwise, m_1 and w_1 would be able to block μ^*. Since μ^* is dynamically stable and $\succ_{m_2} \in \bar{\mathcal{S}}_{m_2}$,

$$w_3w_3 \succ_{m_2} w_3w_1 = \tilde{\mu}(m_2) \succ_{m_2} \mu^*(m_2) \succ_{m_2} w_1w_1$$

for some $w_3 \in W$, $w_3 \neq w_1$. Continuing in this fashion we can define a sequence of distinct men $m_3, m_4, \ldots$ and a sequence of distinct women $w_1, w_3, w_4, \ldots$. However, this is a contradiction as there is but a finite number of agents in the economy. Therefore μ^* must be Pareto optimal. □

Lemma A.3. *If $\succ_i \in \bar{\mathcal{S}}_i$ for all i, then the E-DA and the P-DAA matchings coincide.*

Proof. Let $\tilde{\mu}$ be the interim matching identified in the P-DAA procedure's first step. Three facts are of note:

1. For all $i \neq j$, $\tilde{\mu}(i) \neq ji$. Suppose $\tilde{\mu}(i) = ji$. When $\succ_i \in \bar{\mathcal{S}}_i$, $jj \succ_i \tilde{\mu}(i) = ji$. Likewise, $ii \succ_j \tilde{\mu}(j) = ij$. But this implies i and j can period-1 block $\tilde{\mu}$, which is a contradiction.

2. For all $i \neq j$, $\tilde{\mu}(i) \neq ij$. The argument parallels the preceding case.

3. If $\tilde{\mu}(i) = ii$, then $\mu^*(i) = ii$. Suppose that $\mu^*(i) = ij$, $j \neq i$. However, $\succ_i \in \bar{\mathcal{S}}_i$ implies that $jj \succ_i \tilde{\mu}(i)$. Noting cases (1) and (2), it must be that $\tilde{\mu}(j) = jj$ and thus $ii \succ_i \tilde{\mu}(j)$. Therefore, i and j can period-1 block $\tilde{\mu}$, which is a contradiction.

Facts (1)–(3) imply that $\tilde{\mu}(i) = ii$ or $\tilde{\mu}(i) = jj$ and $\tilde{\mu} = \mu^*$. Clearly, the same matching would be generated if in phase 1 men were restricted to proposals from the set $X_m^0 = \{ww \colon w \in W\}$. This corresponds to men proposing according to $P_{\succ_m}$ as in the E-DA mechanism. □

Proof of Theorem 5. Consider the dynamically stable matching $\mu = (\mu_1, \mu_2)$ and let $\bar{\mu} = (\mu_2, \mu_2)$. If $\mu_1 = \mu_2$, then the theorem is trivially true. Henceforth, suppose $\mu_1 \neq \mu_2$. We argue by contradiction.

Suppose $\bar{\mu}$ can be blocked by agent i alone. If $ii \succ_i \bar{\mu}(i)$ then $\succ_i \in \bar{\mathcal{S}}_i$ implies $(\mu_1(i), i) \succ_i (\mu_1(i), \mu_2(i))$. Thus, μ is not dynamically stable—a contradiction. If instead $(\mu_2(i), i) \succ_i \bar{\mu}(i)$, $\succ_i \in \bar{\mathcal{S}}_i$ implies $ii \succ_i (\mu_2(i), \mu_2(i))$, which is a contradiction.

Suppose m and w can block $\bar{\mu}$. There are four relevant cases:

1. Suppose $ww \succ_m \bar{\mu}(m)$ and $mm \succ_w \bar{\mu}(w)$. This implies that $wP_{\succ_m}\mu_2(m)$ and $mP_{\succ_w}\mu_2(w)$. Hence, $(\mu_1(m), w) \succ_m (\mu_1(m), \mu_2(m))$ and $(\mu_1(w), m) \succ_w (\mu_1(w), \mu_2(w))$. But this implies m and w can period-2 block μ—a contradiction.

2. Suppose $wm \succ_m \bar{\mu}(m)$ and $mw \succ_w \bar{\mu}(w)$. From above, we know that $\bar{\mu}$ is dynamically individually rational. Hence, $\bar{\mu}(m) \succsim_m mm$ and $\bar{\mu}(w) \succsim_w ww$. Noting case (a) above, we may assume that $\bar{\mu}(m) \succ_m mm$ and $\bar{\mu}(w) \succ_w ww$. But this implies $ww \succ_m wm \succ_m \bar{\mu}(m)$ and $mm \succ_w mw \succ_w \bar{\mu}(w)$. Hence, case 1 applies.

3. Suppose $mw \succ_m \bar{\mu}(m)$ and $wm \succ_w \bar{\mu}(w)$. An analogous argument to the preceding case applies.

4. Suppose m and w can period-2 block $\bar{\mu}$. Then $(\mu_2(m), w) \succ_m (\mu_2(m), \mu_2(m)) \implies ww \succ_m \bar{\mu}(m)$ and $(\mu_2(w), m) \succ_w (\mu_2(w), \mu_2(w)) \implies mm \succ_w \bar{\mu}(w)$. Hence, case 1 applies.

In each case above we are led to a contradiction. Hence, $\bar{\mu}_2$ is dynamically stable. □

Proof of Theorem 6. Fix a profile of reported preferences for all agents $i \neq m_1$. Let μ^* be the P-DAA matching when m_1 truthfully reveals his preference $\succ_{m_1}$. Let $\hat{\mu}$ be the P-DAA matching when m_1 states his preference as $\hat{\succ}_{m_1} \neq \succ_{m_1}$. Assume that $\hat{\mu}(m_1) \succ_{m_1} \mu^*(m_1)$.

Suppose that all other agents report preferences $\succ_i$, $i \neq m_1$, that satisfy SIC. First, the same argument verifying that the one-period deferred acceptance algorithm is strategyproof for m_1 leads us to conclude that $\hat{\mu}(m_1) \notin \{w_1w_1, w_1m_1, m_1w_1, m_1m_1\}$ for some $w_1 \in W$. Thus, $\hat{\mu}(m_1) = w_1w_2$ for $w_1 \neq w_2$. To attain this volatile final matching in the P-DAA, m_1 must have announced a preference where $w_1w_2 \mathrel{\hat{\succ}_{m_1}} w_1m_1 \mathrel{\hat{\succ}_{m_1}} m_1m_1$. Since the announced preference must satisfy SIC, SIC(2) implies that $w_2w_2 \mathrel{\hat{\succ}_{m_1}} w_1m_1$. Note that w_1m_1 is m_1's (interim) matching after step 1 of the P-DAA procedure given his report of $\hat{\succ}_{m_1}$.

Now consider w_2. Since she participates in step 2 of the P-DAA, $\hat{\mu}(w_2) = jm_1 \succ_{w_2} jw_2 \succsim_{w_2} w_2w_2$ where jw_2 is her (interim) matching after the procedure's first step and $j \neq m_1$. Since $\succ_{w_2}$ satisfies SIC, SIC(2) implies that $m_1m_1 \succ_{w_2} jw_2$. Since $w_2w_2 \mathrel{\hat{\succ}_{m_1}} w_1m_1$ and $m_1m_1 \succ_{w_2} jw_2$, the interim matching following step 1 of the P-DAA procedure is not ex ante stable with respect to the stated preferences, which is a contradiction. □

Proof of Theorem 7. Recall that $A(e) = \mu = (\mu_1, \mu_2)$ and $\bar{\mu} = (\mu_1, \mu_1)$. Suppose $\mu_1 \neq \mu_2$. Else, the theorem is trivially true. First, we verify that $\bar{\mu}$ is dynamically individually rational. Suppose for some $m \in M$, $mm \succ_m \bar{\mu}(m)$. This implies $\mu_1(m) = w_1 \in W$. If $\mu_2(m) = m$, then $w_1m \succ_m mm \succ_m w_1w_1 = \bar{\mu}(m)$, which is a contradiction as $\succ_m \in \bar{\mathcal{S}}_m$. Therefore, $\mu_2(m) = w_2 \neq w_1$ and hence

$$w_2w_2 \succ_m w_1w_2 = \mu(m) \succ_m mm \succ_m w_1w_1.$$

Now consider an alternative economy, e', with the same agents and where the preferences of all $i \neq m$ are exactly as in e, i.e. $\succ_i = \succ'_i$ for all $i \neq m$. However, the preferences of agent m, $\succ'_m$, are defined as follows: (i) $\succ'_m$ has the same ex ante spot ranking as $\succ_m$, i.e. $P_{\succ'_m} = P_{\succ_m}$; (ii) for all $j \neq k$, $ii \succ'_m jk$; and, (iii) if $i \neq j$ and $k \neq l$, $ij \succ_m kl \iff ij \succ_m kl$. Clearly, $\succ_m \in \bar{\mathcal{S}}_m$. Given these preferences, it follows that $mm \succ'_m w_1w_1 \succ'_m w_1 i$ for all $i \in W_m \backslash \{w_1\}$.

As the procedure A is non-prophetic, $A_1(e) = A_1(e')$. This implies that in the matching $A_1(e')$, agent m is matched to w_1 in period 1. But this contradicts A always generating a dynamically stable matching when agents' preferences are in $\bar{\mathcal{S}}_i$. Thus, $\bar{\mu}(m) \succsim_m mm$. Noting this fact, it follows that $\bar{\mu}(m) \succsim_m (\mu_1(m), m)$ as well. Hence, $\bar{\mu}$ is dynamically individually rational.

Suppose some pair, m and w, can period-1 block $\bar{\mu}$. Since $\bar{\mu}$ is dynamically individually rational, there are three possible cases:

1. Suppose $ww \succ_m (\mu_1(m), \mu_1(m))$ and $mm \succ_w (\mu_1(w), \mu_1(w))$. Clearly, $w \neq \mu_1(m)$ and

$m \neq \mu_1(w)$. Now consider an alternative economy e' where the preferences of all agents other than m and w are identical to those in e. However, the preferences of m, $\succ'_m$, are identical to $\succ_m$ except that all persistent partnership plans are shifted to the very top of the preference ranking and $P_{\succ'_m} = P_{\succ_m}$. (This is analogous to the definition of $\succ'_m$ above.) Define $\succ'_w$ similarly. In this alternative economy, matching mechanism A must assign agent m to $\mu_1(m)$ in period 1. However $ww \succ'_m (\mu_1(m), i)$ for all $i \in W_m$. Likewise, w must be assigned to $\mu_1(w)$, but $mm \succ'_w (\mu_1(w), j)$ for all $j \in M_w$. Hence, m and w would be able to period-1 block the matching generated by A in the economy e', contradicting that A always generates a dynamically stable matching when agents' preferences are in $\bar{\mathcal{S}}_i$.

2. Suppose $mw \succ_m (\mu_1(m), \mu_1(m))$ and $wm \succ_w (\mu_1(w), \mu_1(w))$. Since $\succ_i \in \bar{\mathcal{S}}_i$, $ww \succ_m mw \succ_m (\mu_1(m), \mu_1(m))$ and $mm \succ_w wm \succ_w (\mu_1(w), \mu_1(w))$. Thus, case (1) applies.

3. Suppose $wm \succ_m (\mu_1(m), \mu_1(m))$ and $mw \succ_w (\mu_1(w), \mu_1(w))$. The same reasoning as case (2) and (1) applies.

Therefore, no pair wishes to period-1 block $\bar{\mu}$.

Finally, suppose m and w can period-2 block $\bar{\mu}$. Then $(\mu_1(m), w) \succ_m \bar{\mu}(m) \implies ww \succ_m (\mu_1(m), w) \succ_m \bar{\mu}(m)$. Likewise, $(\mu_1(w), m) \succ_w \bar{\mu}(w) \implies mm \succ_w (\mu_1(w), m) \succ_w \bar{\mu}(w_1)$. But this implies m and w could period-1 block $\bar{\mu}$, which by the previous argument is not possible. Thus, no pair wishes to period-2 block $\bar{\mu}$. □

Proof of Theorem 8. Suppose $\mu(i) = (\mu_1(i), \mu_2(i)) \succ_i (\mu_1(i), \mu_1(i)) = \bar{\mu}(i)$. Since $\succ_i \in \bar{\mathcal{S}}_i$, $(\mu_2(i), \mu_2(i)) \succ_i \mu(i) \succ_i \bar{\mu}(i)$. Clearly, this implies $\mu_2(i) = j \neq i$. Thus, $jj \succ_i \mu(i) \succ_i \bar{\mu}(i)$. Since μ is dynamically stable, $\mu(j) = (\mu_1(j), i) \succ_j ii$. However, $\succ_j \in \bar{\mathcal{S}}_j$, which implies $\bar{\mu}(j) = (\mu_1(j), \mu_1(j)) \succ_j (\mu_1(j), i) = \mu(j)$. □

Lemma A.4. *$\sim_i^{f_i}$ is an equivalence relation.*

Proof. We must verify three properties: reflexivity, symmetry, and transitivity.

1. $\sim_i^{f_i}$ is reflexive. $x \sim_i^{f_i} x' \implies [x \not\succ_i^{f_i} x' \;\&\; x' \not\succ_i^{f_i} x] \implies [x' \not\succ_i^{f_i} x \;\&\; x \not\succ_i^{f_i} x'] \implies x' \sim_i^{f_i} x$.

2. $\sim_i^{f_i}$ is symmetric. Suppose $x \not\sim_i^{f_i} x$. Then $x \succ_i^{f_i} x$. Hence, $\exists \tilde{x} \in f_i(x)$ such that $\forall \tilde{x}' \in f_i(x)$, $\tilde{x} \succ_i \tilde{x}'$. However, $\tilde{x} \in f_i(x)$. Thus, $\tilde{x} \succ_i \tilde{x}$—a contradiction.

3. $\sim_i^{f_i}$ is transitive. Suppose $x \sim_i^{f_i} x'$ and $x' \sim_i^{f_i} x''$. To arrive at a contradiction, suppose $x \succ_i^{f_i} x''$. Thus, $\exists \tilde{x} \in f_i(x)$ such that $\forall \tilde{x}'' \in f_i(x'')$, $\tilde{x} \succ_i \tilde{x}''$. Since $x \sim_i^{f_i} x'$, $\exists \tilde{x}' \in f_i(x')$ such that $\tilde{x} \not\succ_i \tilde{x}'$. This implies $\tilde{x}' \succsim_i \tilde{x}$. But then $\tilde{x}' \succsim_i \tilde{x} \succ_i \tilde{x}''$ for all $\tilde{x}'' \in f_i(x'')$, which implies $x' \succ_i^{f_i} x''$—a contradiction. The same argument applies if instead we assume $x'' \succ_i^{f_i} x$. Hence, $x \sim_i^{f_i} x''$.

□

Remark A.1. The proof of Theorem 9 proceeds similarly to the proof of Theorem 3. The argument is constructive using a generalization of the P-DAA procedure. An analogous generalization of Gale and Shapley's (1962) deferred acceptance algorithm incorporating transfers is the salary-adjustment process of Crawford and Knoer (1981). We apply their intuition to our problem. Therefore, the argument should be familiar.

Definition A.1. The *conditional preference induced by* $\succ_i^{f_i}$ *at* (j, y_1) is defined as $(k, y_2) \succ_i^{(f_i)(j,y_1)} (l, y_2') \iff \left(\begin{smallmatrix} j & k \\ y_1 & y_2 \end{smallmatrix}\right) \succ_i^{f_i} \left(\begin{smallmatrix} j & l \\ y_1 & y_2' \end{smallmatrix}\right)$.

Algorithm A.1 (GP-DAA)**.** The *(man-proposing) generalized plan deferred acceptance procedure with adjustment* identifies an outcome ρ^* as follows. First, for each m define

$$X_m^0 \equiv \left\{ \left(\begin{smallmatrix} w & w \\ y_1 & y_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} w & m \\ y_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} m & w \\ 0 & y_2 \end{smallmatrix}\right) : w \in W; y_1, y_2 \in Y \right\}.$$

Initially, no element of X_m^0 has been rejected. In round $\tau \geq 1$:

1. Let $X_m^\tau \subset X_m^0$ be the set of plans that have not been rejected in some round $\tau' < \tau$. If $X_m^\tau = \emptyset$ or $\left(\begin{smallmatrix} m & m \\ 0 & 0 \end{smallmatrix}\right) \succ_m^{f_m} x$ for all $x \in X_m^\tau$, then m does not make any proposals. In this case, set $\tilde{\rho}^1(m) = \left(\begin{smallmatrix} m & m \\ 0 & 0 \end{smallmatrix}\right)$. Otherwise, let $\left(\begin{smallmatrix} i & j \\ y_1 & y_2 \end{smallmatrix}\right)$ be $\succ_m^{f_m}$-maximal in X_m^τ.[35] If there are several $\succ_m^{f_m}$-maximal elements in X_m^τ, choose a fixed ordering of these elements and propose the first one. (This ordering is maintained through all subsequent rounds, if necessary.) Let $\left(\begin{smallmatrix} i & j \\ y_1 & y_2 \end{smallmatrix}\right)$ be that plan. It can assume one of three forms:
   (a) If $\left(\begin{smallmatrix} i & j \\ y_1 & y_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w & w \\ y_1 & y_2 \end{smallmatrix}\right)$ then m proposes to w a two-period partnership along with the transfer payments (y_1, y_2). From w's perspective, this corresponds to $\left(\begin{smallmatrix} m & m \\ -y_1 & -y_2 \end{smallmatrix}\right)$.
   (b) If $\left(\begin{smallmatrix} i & j \\ y_1 & y_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} w & m \\ y_1 & 0 \end{smallmatrix}\right)$ then m proposes to w a period-1 partnership along with the transfer payments $(y_1, 0)$. From w's perspective, this plan corresponds to $\left(\begin{smallmatrix} m & w \\ -y_1 & 0 \end{smallmatrix}\right)$.

[35] All $\succ_m^{f_m}$-maximal elements involve the same potential partner, but may differ in the profile of transfer payments. They are f_m-equivalent.

(c) If $\left(\begin{smallmatrix} i & j \\ y_1 & y_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} m & w \\ 0 & y_2 \end{smallmatrix}\right)$ then m proposes to w a period-2 partnership along with the transfer payments $(0, y_2)$. From w's perspective, this plan corresponds to $\left(\begin{smallmatrix} w & m \\ 0 & -y_2 \end{smallmatrix}\right)$.

2. Let X^τ_w be the set of plans made available to w. If $\left(\begin{smallmatrix} w & w \\ 0 & 0 \end{smallmatrix}\right) \succ^{f_w}_w x$ for all $x \in X^\tau_m$, w rejects all proposals. Otherwise, if m made the $\succ^{f_w}_w$-maximal in X^τ_w proposal to w, w (tentatively) accepts his proposal and rejects the others.

The above process continues until no further rejections occur. Let $\tilde{\rho}^1$ be the resulting (interim) matching. If $\tilde{\rho}^1_2(i) \neq (i, 0)$, set $\rho^*(i) = \tilde{\rho}^1(i)$, and remove i from the market.

Let $\hat{M}_2$ and $\hat{W}_2$ be the sets of men and women who were not removed from the market in the preceding step. Thus, $\tilde{\rho}^1_2(i) = (i, 0)$ for all $i \in \hat{M}_2 \cup \hat{W}_2$. For these agents define a new period-2 outcome, $\tilde{\rho}^2_2$, as follows. For all $m \in \hat{M}_2$, let $\hat{X}^0_m = \{(w, y_2) \colon w \in \hat{W}_2, y_2 \in Y\}$. Initially no element in $\hat{X}^0_m$ has been rejected. In round $\tau \geq 1$:

1. Let $\hat{X}^\tau_m \subset \hat{X}^0_m$ be the set of period-2 outcomes that have not been rejected in any round $\tau' < \tau$. If $\hat{X}^\tau_m = \emptyset$ or $(m, 0) \succ^{(f_m)(\tilde{\rho}^1_1(m))}_m x$ for all $x \in \hat{X}^\tau_m$, then m does not make any proposals. In this case, set $\tilde{\rho}^2_2(m) = (m, 0)$.

   Otherwise, let (w, y_2) be $\succ^{(f_m)(\tilde{\rho}^1_1(m))}_m$-maximal in $\hat{X}^t_m$. If there are several $\succ^{(f_m)(\tilde{\rho}^1_1(m))}_m$-maximal elements in $\hat{X}^\tau_m$, choose a fixed ordering of these elements and propose the first one. (This ordering is maintained through all subsequent rounds, if necessary.) In this case, m proposes to w a partnership for period 2 along with a period-2 transfer payment of y_2. From w's perspective, that period-2 outcome corresponds to $(m, -y_2)$.

2. Let $\hat{X}^\tau_w$ be the set of period-2 outcomes made available to w. If $(w, 0) \succ^{(f_w)(\tilde{\rho}^1_1(w))}_w x$ for all $x \in \hat{X}^\tau_w$, w rejects all proposals. Otherwise, if m made the $\succ^{(f_w)(\tilde{\rho}^1_1(w))}_w$-maximal in $\hat{X}^\tau_w$ proposal to w, w (tentatively) accepts his proposal and rejects the others.

The above process continues until no further rejections occur. At this point all tentatively-accepted proposals are confirmed. Let $\tilde{\rho}^2_2$ be the resulting (one-period) matching among the agents in $\hat{M}_2 \cup \hat{W}_2$. For all $i \in \hat{M}_2 \cup \hat{W}_2$, set $\rho^*(i) = (\tilde{\rho}^1_1(i), \tilde{\rho}^2_2(i))$.

Lemma A.5. *The interim GP-DAA outcome, $\tilde{\rho}^1$, is ex ante stable.*

Proof. This lemma generalizes Theorem 2. From Definition A.1, it is clear that $\tilde{\rho}^1(i) \succsim^{f_i}_i \left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right)$. Hence, no agent can period-1 block $\tilde{\rho}^1$. If a pair m and w can period-1 block $\tilde{\rho}^1$ then there must exist a plan for m and a corresponding plan for w, that both m and w *strictly* prefer (given f_i) to their assignment under $\tilde{\rho}^1$. However, this implies that m must have proposed that plan to w at some stage of the procedure and she must have rejected it. Hence, she must prefer her assignment under $\tilde{\rho}^1$ to that proposal, which is a contradiction. □

Proof of Theorem 9. We show that the GP-DAA outcome, ρ^*, is dynamically stable. Let $\tilde{\rho}^1$ be the interim matching from the procedure's first step. By standard reasoning, $\rho^*(i) \succsim_i^{f_i} \tilde{\rho}^1(i)$ for all i. Step 2 in the GP-DAA weakly improves agents' assignment. Since $\tilde{\rho}^1$ is ex ante stable (Lemma A.5), ρ^* must also be ex ante stable. Therefore, it cannot be period-1 blocked by any agent or by any pair.

Suppose agent i can period-2 block ρ^*, i.e. $(\rho_1^*(i), (i,0)) \succ_i^{f_i} \rho^*(i)$. Thus, $\rho_1^*(i) \neq (i,0)$ and $\rho_2^*(i) \neq (i,0)$; else, we would arrive at a contradiction. Let $\rho_1^*(i) = (j, y_1)$ for some $j \neq i$. There are two cases. First, if $\tilde{\rho}^1(i) = \left(\begin{smallmatrix} j & j \\ y_1 & y_2 \end{smallmatrix}\right)$, then by G-SIC(1), $\left(\begin{smallmatrix} j & i \\ y_1 & 0 \end{smallmatrix}\right) \succ_i^{f_i} \rho^*(i) \succsim_i^{f_i} \tilde{\rho}^1(i) = \left(\begin{smallmatrix} j & j \\ y_1 & y_2 \end{smallmatrix}\right) \implies \left(\begin{smallmatrix} i & i \\ 0 & 0 \end{smallmatrix}\right) \succ_i^{f_i} \tilde{\rho}^1(i)$, which is also a contradiction. If instead, and second, $\tilde{\rho}^1(i) = \left(\begin{smallmatrix} j & i \\ y_1 & 0 \end{smallmatrix}\right)$, then $\left(\begin{smallmatrix} j & i \\ y_1 & 0 \end{smallmatrix}\right) \succ_i^{f_i} \rho^*(i) \succsim_i^{f_i} \tilde{\rho}^1(i) = \left(\begin{smallmatrix} j & i \\ y_1 & 0 \end{smallmatrix}\right)$, which is a contradiction. Therefore, agent i cannot period-2 block ρ^*.

Suppose m and w can period-2 block ρ^*. This implies that there exists $y_2' \in Y$ such that $\left(\begin{smallmatrix} \mu_1^*(m) & w \\ \sigma_1^*(m) & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \rho^*(m) \succsim_m^{f_m} \tilde{\rho}^1(m)$ and $\left(\begin{smallmatrix} \mu_1^*(w) & m \\ \sigma_1^*(w) & -y_2' \end{smallmatrix}\right) \succ_w^{f_w} \rho^*(w) \succsim_w^{f_w} \tilde{\rho}^1(w)$. If $\mu_1^*(m) = w$ or $\mu_1^*(m) = m$ and $\mu_1^*(w) = w$, then m and w can period-1 block $\tilde{\rho}^1$, which is not possible. Thus, without loss of generality, there are two cases to consider.

1. Suppose $\rho_1^*(m) = (m,0)$ and $\rho_1^*(w) = (m', y_1')$, $m' \neq m$. Thus, $\left(\begin{smallmatrix} m & w \\ 0 & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \rho^*(m) \succsim_m^{f_m} \left(\begin{smallmatrix} m & j \\ 0 & y_2 \end{smallmatrix}\right) = \tilde{\rho}^1(m)$ for some (j, y_2). If $(j, y_2) = (m, 0)$, then G-SIC(1) implies that $\left(\begin{smallmatrix} w & w \\ y_2' & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \left(\begin{smallmatrix} m & m \\ 0 & 0 \end{smallmatrix}\right) = \tilde{\rho}^1(m)$. If instead $(j, y_2) = (w', y_2'')$, $w' \neq m$ then G-SIC(3) implies that $\left(\begin{smallmatrix} w & w \\ y_2' & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \left(\begin{smallmatrix} m & w' \\ 0 & y_2'' \end{smallmatrix}\right) = \tilde{\rho}^1(m)$.

   Similarly, $\left(\begin{smallmatrix} m' & m \\ y_1' & -y_2' \end{smallmatrix}\right) \succ_w^{f_w} \rho^*(w) \succsim_w^{f_w} \left(\begin{smallmatrix} m' & k \\ y_1' & y_2'' \end{smallmatrix}\right) = \tilde{\rho}^1(m)$ for some (k, y_2''). If $(k, y_2'') = (w, 0)$, then G-SIC(2) implies that $\left(\begin{smallmatrix} m & m, \\ -y_2' & -y_2' \end{smallmatrix}\right) \succ_w^{f_w} \left(\begin{smallmatrix} m' & w \\ y_1' & 0 \end{smallmatrix}\right) = \tilde{\rho}^1(w)$. If instead $(k, y_2'') = (m', y_2'')$, then G-SIC(1) implies that $\left(\begin{smallmatrix} m & m, \\ -y_2' & -y_2' \end{smallmatrix}\right) \succ_w^{f_w} \left(\begin{smallmatrix} m' & m' \\ y_1' & y_2'' \end{smallmatrix}\right) = \tilde{\rho}^1(w)$. In each case, m and w can period-1 block $\tilde{\rho}^1$, which is a contradiction.

2. Suppose $\rho_1^*(m) = (w', y_1)$, $w \neq w'$, and $\rho_1^*(w) = (m', y_1')$, $m' \neq m$. Thus, $\left(\begin{smallmatrix} w' & w \\ y_1 & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \rho^*(m) \succsim_m^{f_m} \left(\begin{smallmatrix} w' & j \\ y_1 & y_2 \end{smallmatrix}\right) = \tilde{\rho}^1(m)$ for some (j, y_2). If $(j, y_2) = (m, 0)$, then G-SIC(2) implies that $\left(\begin{smallmatrix} w & w \\ y_2' & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \left(\begin{smallmatrix} w' & m \\ y_1 & 0 \end{smallmatrix}\right) = \tilde{\rho}^1(m)$. If instead $(j, y_2) = (w', y_2''')$, then G-SIC(1) implies that $\left(\begin{smallmatrix} w & w \\ y_2' & y_2' \end{smallmatrix}\right) \succ_m^{f_m} \left(\begin{smallmatrix} w' & w' \\ y_1 & y_2''' \end{smallmatrix}\right) = \tilde{\rho}^1(m)$. Analogously, we conclude that $\left(\begin{smallmatrix} m & m \\ -y_2' & -y_2' \end{smallmatrix}\right) \succ_w^{f_w} \tilde{\rho}^1(w)$. Thus, m and w can period-1 block $\tilde{\rho}^1$, which is a contradiction.

□

B A T-Period Matching Market

This appendix presents a T-period version of Theorem 3. Kadam and Kotowski (2015) study a special case of the model below.

Let M and W be disjoint, finite sets of men and women, respectively. A *partnership plan* for $m \in M$ is a sequence of partners $x = (x_1, x_2, \ldots, x_T) \in W_m^T$. A plan for $w \in W$ is defined analogously. When confusion is unlikely, we will abbreviate a partnership plan as $x = x_1 x_2 \cdots x_T$. The truncation of plan x to the first $t-1$ periods is $x_{<t} = x_1 \cdots x_{t-1}$. Its continuation from period t is $x_{\geq t} = x_t x_{t+1} \cdots x_T$. Definitions of $x_{\leq t}$ and $x_{>t}$ follow analogously. When we write $x = (x_{<t-1}, j, k, x_{>t})$, then j is the specified period-$(t-1)$ partner and k is the period-t partner. We let x^{jk} stand for a partnership plan where $x_t \in \{j, k\}$ for all t. Of course, $x^i = i \cdots i$ is a constant plan. Each agent has a strict and rational preference, $\succ_i$, defined over partnership plans.

Definition B.1. The function $\mu_t \colon M \cup W \to M \cup W$ is a *one-period matching* if

1. For all $m \in M$, $\mu_t(m) \in W_m$;
2. For all $w \in W$, $\mu_t(w) \in M_w$; and,
3. For all i, $\mu_t(i) = j \implies \mu_t(j) = i$.

A *matching*, $\mu \colon M \cup W \to (M \cup W)^T$ is a sequence of one-period matchings, i.e. $\mu = (\mu_1, \ldots, \mu_T)$.

To define blocking and stability, we adopt a coalition-based nomenclature, which we also employ in our analysis of the core (Appendix C). A *coalition* C is a non-empty subset of agents, $C \subset M \cup W$. A coalition can block a matching if it can define a within-coalition matching that its members prefer.

Definition B.2. The function $\mu_t^C \colon C \to C$ is a *one-period matching for coalition C* if

1. For all $m \in M \cap C$, $\mu_t^C(m) \in W_m \cap C$;
2. For all $w \in W \cap C$, $\mu_t^C(w) \in M_w \cap C$; and,
3. For all $i \in C$, $\mu_t^C(i) = j \implies \mu_t^C(j) = i$.

Definition B.3. A coalition C can *period-t block* the matching μ if there exists a sequence of one-period matchings for the coalition, $\mu_{\geq t}^C = (\mu_t^C, \mu_{t+1}^C, \ldots, \mu_T^C)$ such that $(\mu_{<t}(i), \mu_{\geq t}^C(i)) \succ_i \mu(i)$ for all $i \in C$.

Definition B.4. A coalition C is *admissible* if (i) $C = \{i\}$ for some $i \in M \cup W$ or (ii) $C = \{m, w\}$ for some $m \in M$ and $w \in W$.

Definition B.5. The matching μ is *ex ante stable* if it cannot be period-1 blocked by any admissible coalition.

Definition B.6. The matching μ is *dynamically stable* if for all t it cannot be period-t blocked by any admissible coalition.

To prove the existence of dynamically stable matching, we rely on a constructive proof using a generalization of the P-DAA procedure. The following mechanism reduces to the Gale and Shapley (1962) deferred acceptance algorithm when $T = 1$ and to the P-DAA procedure when $T = 2$.

Algorithm B.1 (TP-DAA)**.** The *T-period (man-proposing) plan deferred acceptance procedure with adjustment* identifies a matching μ^* in a series of steps as follows:

Step 1. For each m let

$$X_m^0 = \bigcup_{w \in W} \Big\{(i_1, \ldots, i_T) \colon i_t \in \{m, w\}\Big\} \setminus \{x^m\}.$$

At $\tau = 0$, no plans in X_m^0 have been rejected. In round $\tau \geq 1$:

1. Let $X_m^\tau \subset X_m^0$ be the subset of plans that have not been rejected in some round $\tau' < \tau$. If $X_m^\tau = \emptyset$ or $x^m \succ_m x$ for all $x \in X_m^\tau$, then m does not make any proposals. Otherwise, m proposes to the woman identified in his most preferred plan in X_m^τ. Each such plan involves at most one distinct woman. If x is his most preferred plan and it involves w, he proposes to w the corresponding plan. For example, if $x = wwmw\cdots$, then m proposes to w a plan where they are paired for periods 1 and 2, unmatched in period 3, matched together in period 4, and so on.

2. Let X_w^τ be the set of plans made available to w. If $x^w \succ_w x$ for all $x \in X_w^\tau$, w rejects all proposals. Otherwise, w (tentatively) accepts her most preferred plan in X_w^τ and rejects the others.

The above process continues until no rejections occur. If w accepts m's proposal in the final round, define the interim matchings $\tilde{\mu}^1(m)$ and $\tilde{\mu}^1(w)$ accordingly. If i does not make or receive any proposals in the final round, set $\tilde{\mu}^1(i) = i \cdots i$.

Step $s \geq 2$. If $\tilde{\mu}^{s-1}_{\geq s}(i) \neq x^i_{\geq s}$, set $\tilde{\mu}^s(i) = \tilde{\mu}^{s-1}(i)$. Else, let $\hat{M}_s = \{m \in M : \tilde{\mu}^{s-1}_{\geq s}(m) = x^m_{\geq s}\}$ and $\hat{W}_s = \{w \in W : \tilde{\mu}^{s-1}_{\geq s}(w) = x^w_{\geq s}\}$ be the sets of the remaining men and women, respectively. Each man (woman) in $\hat{M}_s$ ($\hat{W}_s$) is unmatched in each period $t \geq s$ under the interim matching $\tilde{\mu}^{s-1}$.

For each $m \in \hat{M}_s$ let

$$X^0_m = \bigcup_{w \in \hat{W}_s} \Big\{(i_s, i_{s+1}, \ldots, i_T) : i_t \in \{m, w\}\Big\} \setminus \{x^m_{\geq s}\}.$$

be a set of candidate continuation plans for m. At $\tau = 0$, no continuation plans in X^0_m have been rejected. In round $\tau \geq 1$:

1. Let $X^\tau_m \subset X^0_m$ be the subset of plans that have not been rejected in some round $\tau' < \tau$. If $X^\tau_m = \emptyset$ or $(\tilde{\mu}^{s-1}_{<s}(m), x^m_{\geq s}) \succ_m (\tilde{\mu}^{s-1}_{<s}(m), x_{\geq s})$ for all $x_{\geq s} \in X^\tau_m$, then m does not make any proposals. Otherwise, let $x^{mw}_{\geq s} \in X^\tau_m$ be m's most preferred, not yet rejected continuation plan, i.e.

$$(\tilde{\mu}^{s-1}_{<s}(m), x^{mw}_{\geq s}) \succ_m (\tilde{\mu}^{s-1}_{<s}(m), x_{\geq s})$$

for all $x_{\geq s} \in X^\tau_m \setminus \{x^{mw}_{\geq s}\}$. This plan involves exactly one woman, in this case w. He proposes to her a compatible continuation plan. For example, if $x^{mw}_{\geq s} = (w, w, m, w, \ldots)$ then m proposes to w a continuation plan where they are paired in periods s and $s+1$, unmatched in period $s+2$, matched together in period $s+3$, and so on.

2. Let $X^\tau_w = \{x^{mw}_{\geq s}, x^{m'w}_{\geq s}, \ldots\}$ be the set of continuation plans made available to w. If $(\tilde{\mu}^{s-1}_{<s}(m), x^w_{\geq s}) \succ_w (\tilde{\mu}^{s-1}_{<s}(w), x_{\geq s})$ for all $x_{\geq s} \in X^\tau_w$, w rejects all proposals. Otherwise, w (tentatively) accepts her most preferred continuation plan in X^τ_w and rejects the others.

The above process continues until no rejections occur. If w accepts m's proposal in the final round, say $x^{mw}_{\geq s}$, set $\tilde{\mu}^s(w) = (\tilde{\mu}^{s-1}_{<s}(w), x^{mw}_{\geq s})$. Define $\tilde{\mu}^s(m)$ in a corresponding fashion accordingly. If i does not make or receive any proposals in the final round, set $\tilde{\mu}^s(i) = (\tilde{\mu}^{s-1}_{<s}(i), x^i_{\geq s})$.

Final Assignment. Let $\tilde{\mu}^T$ be the matching identified at the conclusion of step $s = T$. Let this be the final matching, i.e. $\mu^* = \tilde{\mu}^T$.

Lemma B.1. *There exists an ex ante stable matching.*

Proof. Step 1 of the TP-DAA procedure provides a T-period generalization of the plan deferred acceptance procedure. It is straightforward to verify that the matching identified by step 1 of the TP-DAA procedure, $\tilde{\mu}^1$, is ex ante stable. The argument mirrors the reasoning of the two-period case. □

In the two-period case, a restriction on agents' preferences was required to ensure the existence of a dynamically stable matching. The following is a T-period generalization of that restriction.

Definition B.7 (T-SIC)**.** The preference $\succ_i$ satisfies *T-period sequential improvement complementarity* if for all $t \geq 2$, all $x = x_1 \cdots x_T$, all x^{ik}, and all x^{ij},

1. $(x_{<t-1}, j, x^{ik}_{\geq t}) \succ_i (x_{<t-1}, j, x^{ij}_{\geq t}) \succsim_i x^i$ implies that

$$(x_{<t-1}, k, x^{ik}_{\geq t}) \succ_i (x_{<t-1}, j, x^{ij}_{\geq t}).$$

2. $x^{ik}_{\geq t} \neq x^i_{\geq t}$ and $(x^i_{<t-1}, j, x^{ik}_{\geq t}) \succ_i (x^i_{<t-1}, j, x^i_{\geq t}) \succ_i x^i$ imply that

$$(x^i_{<t-1}, k, x^{ik}_{\geq t}) \succ_i (x^i_{<t-1}, j, x^i_{\geq t}).$$

3. $x^{ij}_{\geq t} \neq x^i_{\geq t}$ and $(x^i_{<t-1}, i, x^{ik}_{\geq t}) \succ_i (x^i_{<t-1}, i, x^{ij}_{\geq t}) \succ_i x^i$ imply that

$$(x^i_{<t-1}, k, x^{ik}_{\geq t}) \succ_i (x^i_{<t-1}, i, x^{ij}_{\geq t}).$$

Theorem B.1. *If agents' preferences satisfy T-SIC, there exists a dynamically stable matching.*

Proof. The following argument extends by induction the reasoning from the two-period case. Let μ^* be the matching identified by the TP-DAA procedure. Let $(\tilde{\mu}^1, \ldots, \tilde{\mu}^T)$ be the sequence of interim matchings identified during the procedure's operation. Each step of the TP-DAA weakly improves the interim matching, i.e. $\mu^*(i) \succsim_i \tilde{\mu}^{t+1}(i) \succsim_i \tilde{\mu}^t(i)$ for all i and $t < T$. Hence, if μ^* can be period-t blocked by an agent or a pair, then the interim matching $\tilde{\mu}^t$ can be period-t blocked by the same agent or pair. This is because the TP-DAA ensures that $\mu^*_{<t} = \tilde{\mu}^t_{<t}$. Therefore, to verify that μ^* is dynamically stable, it is sufficient to show that $\tilde{\mu}^t$ cannot be period-t blocked for each t.

By Lemma B.1, $\tilde{\mu}^1$ is ex ante stable and cannot be period-1 blocked by any agent or by any pair. Proceeding by induction, suppose $\tilde{\mu}^{t'}$ cannot be blocked by any agent or pair in period $t' = t - 1$. We will verify that $\tilde{\mu}^t$ cannot be blocked in period t.

As a preliminary observation clarifying notation, from the TP-DAA procedure we know that

$$\tilde{\mu}^t(i) = (\tilde{\mu}^{t-1}_{<t}(i), x^{ij'}_{\geq t}) = (\overbrace{\tilde{\mu}^{t-1}_{<t-1}(i), \underset{\substack{\uparrow \\ t-1}}{j}}^{\tilde{\mu}^{t-1}_{<t}(i)}, x^{ij'}_{\geq t}) \tag{B.1}$$

for some j and j'.

Suppose agent i can period-t block $\tilde{\mu}^t$. Then

$$(\tilde{\mu}^{t-1}_{<t-1}(i), j, x^i_{\geq t}) \succ_i \tilde{\mu}^t(i) \succsim_i \tilde{\mu}^{t-1}(i) \succsim_i x^i. \tag{B.2}$$

First, note that $j \neq i$. Otherwise, $(\tilde{\mu}^{t-1}_{<t}(i), i, x^i_{\geq t}) \succ_i \tilde{\mu}^t(i) \succsim_i \tilde{\mu}^{t-1}(i)$. This implies that $\tilde{\mu}^{t-1}$ can be period-$(t-1)$ blocked by i, contradicting the induction hypothesis. Since $j \neq i$, it follows that $\tilde{\mu}^{t-1}(i) = (\tilde{\mu}^{t-1}_{<t-1}(i), x^{ij}_{\geq t-1}) = (\tilde{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t})$ for some j. Applying T-SIC(1) with $k = i$ gives

$$\begin{aligned} &(\tilde{\mu}^{t-1}_{<t-1}(i), j, x^i_{\geq t}) \succ_i (\tilde{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t}) \succsim_i x^i \\ &\quad \Longrightarrow (\tilde{\mu}^{t-1}_{<t-1}(i), i, x^i_{\geq t}) \succ_i (\tilde{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t}) = \tilde{\mu}^{t-1}(i). \end{aligned}$$

Hence, $\tilde{\mu}^{t-1}$ can be period-$(t-1)$ blocked by i, which is a contradiction.

Suppose instead that m and w can period-t block $\tilde{\mu}^t$. Then there exists a sequence of matchings among m and w,

$$\mu^{mw}_{\geq t} = (\mu^{mw}_t, \ldots, \mu^{mw}_T) \tag{B.3}$$

such that

$$(\tilde{\mu}^{t-1}_{<t-1}(m), \tilde{\mu}^{t-1}_{t-1}(m), \mu^{mw}_{\geq t}(m)) \succ_m \tilde{\mu}^t(m) \succsim_m \tilde{\mu}^{t-1}(m) \succsim_m x^m \tag{B.4}$$

and

$$(\tilde{\mu}^{t-1}_{<t-1}(w), \tilde{\mu}^{t-1}_{t-1}(w), \mu^{mw}_{\geq t}(w)) \succ_w \tilde{\mu}^t(w) \succsim_w \tilde{\mu}^{t-1}(w) \succsim_w x^w. \tag{B.5}$$

If $\tilde{\mu}^{t-1}_{t-1}(m) = m$ and $\tilde{\mu}^{t-1}_{t-1}(w) = w$, then m and w can period-$(t-1)$ block $\tilde{\mu}^{t-1}$, which is a contradiction. The same conclusion applies if $\tilde{\mu}^{t-1}_{t-1}(m) = w$ and $\tilde{\mu}^{t-1}_{t-1}(m) = m$. Thus, without loss of generality, there are two cases.

1. Suppose $\tilde{\mu}^{t-1}_{t-1}(m) = m$ and $\tilde{\mu}^{t-1}_{t-1}(w) = m' \neq w$. Consider first agent m. It follows that

$$(\tilde{\mu}^{t-1}_{<t-1}(m), m, \mu^{mw}_{\geq t}(m)) \succ_m \tilde{\mu}^{t-1}(m) = (\tilde{\mu}^{t-1}_{<t-1}(m), m, x^{mj}_{\geq t}) \succsim_i x^m$$

for some $j \in \{m, w'\}$. If $j = m$, then $x^{mj}_{\geq t} = x^{m}_{\geq t}$. T-SIC(1) implies that

$$(\tilde{\mu}^{t-1}_{<t-1}(m), w, \mu^{mw}_{\geq t}(m)) \succ_m \tilde{\mu}^{t-1}(m) = (\tilde{\mu}^{t-1}_{<t-1}(m), m, x^{m}_{\geq t}). \tag{B.6}$$

If $j = w' \in W_m$, then $x^{mj}_{\geq t} \neq x^{m}_{\geq t}$. T-SIC(3) implies that

$$(\tilde{\mu}^{t-1}_{<t-1}(m), w, \mu^{mw}_{\geq t}(m)) \succ_m \tilde{\mu}^{t-1}(m) = (\tilde{\mu}^{t-1}_{<t-1}(m), m, x^{mw'}_{\geq t}). \tag{B.7}$$

For w,

$$(\tilde{\mu}^{t-1}_{<t-1}(w), m', \mu^{mw}_{\geq t}(w)) \succ_w \tilde{\mu}^{t-1}(w) = (\tilde{\mu}^{t-1}_{<t-1}(w), m', x^{wk}_{\geq t}) \succsim_i x^{w}$$

for $k \in \{w, m'\}$. If $k = w$, then T-SIC(2) implies that

$$(\tilde{\mu}^{t-1}_{<t-1}(w), w, \mu^{mw}_{\geq t}(w)) \succ_w \tilde{\mu}^{t-1}(w) = (\tilde{\mu}^{t-1}_{<t-1}(w), m', x^{w}_{\geq t}). \tag{B.8}$$

If instead $k = m'$, then T-SIC(1) implies that

$$(\tilde{\mu}^{t-1}_{<t-1}(w), w, \mu^{mw}_{\geq t}(w)) \succ_w \tilde{\mu}^{t-1}(w) = (\tilde{\mu}^{t-1}_{<t-1}(w), m', x^{wm'}_{\geq t}). \tag{B.9}$$

Together, (B.6)–(B.9) imply that m and w can period-$(t-1)$ block the matching $\tilde{\mu}^{t-1}$ with the continuation plan where m and w are matched together in period $t-1$ and $\mu^{mw}_{\geq t}$ as defined in (B.3) is implemented in periods $t' \geq t$. This is a contradiction since, by the induction hypothesis, $\tilde{\mu}^{t-1}$ cannot be period-$(t-1)$ blocked.

2. Suppose $\tilde{\mu}^{t-1}_{t-1}(m) = w' \neq w$ and $\tilde{\mu}^{t-1}_{t-1}(w) = m' \neq m$. Arguments parallel to cases (B.8) and (B.9) above show that m and w will be able to define a continuation plan to block $\tilde{\mu}^{t-1}$ in period $t-1$, which is a contradiction.

Therefore, $\tilde{\mu}^{t}$ cannot be period-t blocked by m and w. □

Remark B.1. Though the intuition is suggestive, the above analysis does not apply when $T = \infty$. For example, the TP-DAA procedure may fail to terminate. The $T = \infty$ case can be accommodated with further restrictions on preferences.

C The Core

In this appendix we investigate stronger versions of our stability concepts by allowing collective blocking actions. A *coalition* C is a non-empty subset of agents, $C \subset M \cup W$. A coalition can block a matching if it can define a within-coalition matching that its members prefer. A core matching is immune to such collective deviations. More formally, we have the following analogues of previously introduced ideas.

Definition C.1. The function $\mu_t^C : C \to C$ is a *one-period matching for coalition* C if

1. For all $m \in M \cap C$, $\mu_t^C(m) \in W_m \cap C$;
2. For all $w \in W \cap C$, $\mu_t^C(w) \in M_w \cap C$; and,
3. For all $i \in C$, $\mu_t^C(i) = j \implies \mu_t^C(j) = i$.

Definition C.2. A coalition C can *period-1 block* the matching μ if there exist one-period matchings for the coalition C, μ_1^C and μ_2^C, such that $(\mu_1^C(i), \mu_2^C(i)) \succ_i \mu(i)$ for all $i \in C$.

Definition C.3. A coalition C can *period-2 block* the matching μ if there exists a one-period matching for coalition C, μ_2^C, such that $(\mu_1(i), \mu_2^C(i)) \succ_i \mu(i)$ for all $i \in C$.

Definition C.4. The matching μ is in the *ex ante core* if it cannot be period-1 blocked by any coalition.

Definition C.5. The matching μ is in the *dynamic core* if for all t it cannot be period-t blocked by any coalition.

The definitions of the ex ante core and of the dynamic core reduce to those of ex ante and dynamic stability, respectively, when only one-agent or couple coalitions are allowed. The ex ante core corresponds to the "core" in Damiano and Lam (2005, Definition 3). What we call the dynamic core is sometimes called the "recursive core" (Damiano and Lam, 2005, Definition 4; Becker and Chakrabarti, 1995). The dynamic core differs from "dynamic group-stability," studied by Kurino (2009), which allows deviating agents to be matched with non-coalition members.

In a one-period market, the core is not empty and corresponds to the set of pairwise stable matchings (Gale and Shapley, 1962). Several studies of dynamic matching markets have noted the core's emptiness (Damiano and Lam, 2005; Kurino, 2009). Similarly, the core may be empty in static many-to-many matching models (Blair, 1988). In our setting, both the ex ante core and the dynamic core can be empty, even when $\succ_i \in \bar{\mathcal{S}}_i$ for all i. We define the class of preferences $\bar{\mathcal{S}}_i$ in the main text.

Example C.1. Consider the following economy with three men and three women:

$$\succ_{m_1}: w_2w_2, w_3w_3, w_3w_2, m_1w_2, w_1w_2, w_1w_3, m_1m_1$$
$$\succ_{m_2}: w_3w_3, w_1w_1, w_1w_3, m_2w_3, w_2w_3, w_2w_1, m_2m_2$$
$$\succ_{m_3}: w_1w_1, w_2w_2, w_2w_1, m_3w_1, w_3w_1, w_3w_2, m_3m_3$$

$$\succ_{w_1}: m_1m_1, m_1w_1, m_1m_2, m_1m_3, w_1w_1$$
$$\succ_{w_2}: m_2m_2, m_2w_2, m_2m_3, m_2m_1, w_2w_2$$
$$\succ_{w_3}: m_3m_3, m_3w_3, m_3m_1, m_3m_2, w_3w_3$$

In this case, $\succ_i \in \bar{\mathcal{S}}_i$ for all i. There are four ex ante stable matchings (Table C.1). Each can be blocked by some coalition. Since all matchings in the ex ante and the dynamic cores must be ex ante stable, the ex ante and dynamic cores are empty.

Table C.1: All ex ante stable matchings in Example C.1 and blocking coalitions.

Matching	m_1	m_2	m_3	w_1	w_2	w_3	Blocking Coalition
μ^1	w_1w_2	w_2w_1	m_3m_3	m_1m_2	m_2m_1	w_3w_3	$\{m_2, m_3, w_2, w_3\}$
μ^2	m_1m_1	w_2w_3	w_3w_2	w_1w_1	m_2m_3	m_3m_2	$\{m_1, m_3, w_1, w_3\}$
μ^3	w_1w_3	m_2m_2	w_3w_1	m_1m_3	w_2w_2	w_3w_1	$\{m_1, m_2, w_1, w_2\}$
μ^4	m_1m_1	m_2m_2	m_3m_3	w_1w_1	w_2w_2	w_3w_3	$\{m_2, m_3, w_2, w_3\}$

To ensure the core's non-emptiness, we introduce several new classes of preferences. Noting that core and stable matchings coincide in Gale and Shapley's (1962) model, our first restriction moves our dynamic economy closer to their static setting by strengthening the degree of inertia in agents' preferences.

Definition C.6. Let $\succ_i$ be a preference for agent i.

1. $\succ_i$ exhibits *strong inertia* if for all j, k, $j \neq k$, $jj \succ_i jk$ and $jj \succ_i kj$. Let $\mathcal{I}_i$ denote the set of preferences for i that exhibit strong inertia.

2. $\succ_i$ exhibits *very strong inertia* if $jj \succ_i kl$ for all j, k, l, $k \neq l$. Let $\mathcal{I}_i^*$ denote the set of preferences for i that exhibit very strong inertia.

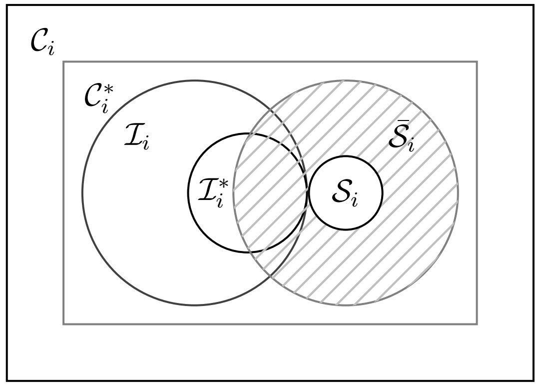

Figure C.1: Preference domains. $\mathcal{S}_i$ – preferences that reflect a spot ranking; $\bar{\mathcal{S}}_i$ – preferences that exhibit inertia relative to $\mathcal{S}_i$; $\mathcal{I}_i$ – preferences with strong inertia; $\mathcal{I}_i^*$ – preferences with very strong inertia; $\mathcal{C}_i^*$ – preferences that exhibit sequential local improvement complementarity; $\mathcal{C}_i$ – preferences that exhibit sequential improvement complementarity.

As clear from Definition C.6 when $\succ_i \in \mathcal{I}_i^*$ for all i, our model reduces to an essentially static setting. Similarly, when $\succ_i \in \mathcal{I}_i$ for all i, all stable matchings are persistent. In both cases the set of stable matchings will coincide with the core.

While preferences with strong and very strong inertia offer a simple route to positive results, they may be too strong for applications. Even if the preferences of one side of the market exhibit strong inertia, it may be possible to place fewer restrictions on the preferences of agents on the market's other side. We propose the following condition.

Definition C.7 (SLIC)**.** The preference $\succ_i$ satisfies *sequential local improvement complementarity* if

1. $jk \succ_i ll' \succsim_i jj \succsim_i ii \implies kk \succsim_i ll'$
2. $jk \succ_i ji \succ_i ii \implies kk \succ_i ji$; and,
3. $ik \succ_i ij \succ_i ii \implies kk \succ_i ij$.

Sequential local improvement complementarity resembles sequential improvement complementarity with the exception of its first restriction.[36] If j and k are complements, i.e. $jk \succ_i \{jj, kk\}$, then sequential local improvement complementarity requires that (i) $jk \succ_i kk \succ_i \ldots \succ_i jj$ and (ii) i does not rank any other plans "in between" jk and kk.

[36]Condition 1 implies conditions 2 and 3. We include these conditions in the definition of SLIC to emphasize the parallel with SIC.

If $\mathcal{C}_i$ is the set of preferences satisfying SIC, we let $\mathcal{C}_i^*$ be the set of preferences satisfying SLIC. It follows that $\bar{\mathcal{S}}_i \cup \mathcal{I}_i \subset \mathcal{C}_i^* \subset \mathcal{C}_i$. Figure C.1 illustrates the relationships among the introduced preference domains.

Theorem C.1. *Suppose $\succ_m \in \mathcal{C}_m^*$ for all $m \in M$ and $\succ_w \in \mathcal{I}_w$ for all $w \in W$. The dynamic core is not empty and coincides with the set of (pairwise) dynamically stable matchings.*

Proof. Since $\mathcal{C}_i^* \subset \mathcal{C}_i$ and $\mathcal{I}_i \subset \mathcal{C}_i$, there exists a dynamically stable matching, say μ^*. Suppose μ^* can be blocked by coalition C in period 1. Let $w \in C$. Suppose $\mu_t^C(w) = w$ for some t. Since $\succ_w \in \mathcal{I}_w$, $ww \succsim_w \mu_t^C(w) \succ_w \mu^*(w)$, which is a contradiction. Thus, $\mu^C(w) = m'm$, which implies $\{mm, m'm'\} \succsim_w \mu^C(w) \succ_w \mu^*(w)$.

Consider agent $m \in M$. Since $mm \succ_w \mu^*(w)$ and μ^* is dynamically stable, $\mu^*(m) \succ_m ww$. If $\mu^C(m) = mw$, then $mw \succ_m \mu^*(m) \succ_m mm$ implies that $ww \succ_m \mu^*(m)$. But then m and w can period-1 block μ^*, which is not possible. If instead $\mu^C(m) = w'w$, then since $mm \succ_{w'} \mu^C(w') \succ_{w'} \mu^*(w')$, it follows that $w'w \succ_m \mu^*(m) \succ_m w'w'$. But since $\succ_m \in \mathcal{C}_m^*$, this implies that $ww \succ_m \mu^*(m)$. Again, this is a contradiction. Therefore, no coalition can period-1 block μ^*.

Suppose μ^* can be blocked by coalition C in period 2. Let $w \in C$. If $\mu_2^C(w) = w$, then $ww \succ_w \mu^*(w)$, which is a contradiction. Thus, there exists $m \in C \cap M$ such that $\mu_2^C(w) = m$ and $mm \succsim_w (\mu_1^*(w), \mu_2^C(w)) \succ_w \mu^*(w)$. This implies $(\mu_1^*(m), w) \succ_m \mu^*(m) \succ_m ww$. There are two cases. If $\mu_1^*(m) = m$, then $\succ_m \in \mathcal{C}_m^*$ implies that $ww \succ_m \mu^*(m)$, which is a contradiction. If instead $\mu_1^*(m) = w'$, then $w'w \succ_m \mu^*(m) \succ_m w'w'$, which again implies that $ww \succ_m \mu^*(m)$—a contradiction.

Thus, no coalition can block μ^* in period 1 or period 2 and μ^* is in the dynamic core. ☐

The following example shows that Theorem C.1 does not obtain if the restriction on men's preference is relaxed to $\succ_m \in \mathcal{C}_m$ for all $m \in M$.

Example C.2. Consider a market with two men and two women whose preferences are

$$\succ_{m_1}: w_1w_2, w_3w_1, m_1m_1 \qquad \succ_{w_1}: m_1m_1, m_2m_2, m_3m_3, m_1m_2, m_3m_1, w_1w_1$$
$$\succ_{m_2}: w_3w_2, w_2w_1, m_2m_2 \qquad \succ_{w_2}: m_1m_1, m_2m_2, m_3m_3, m_3m_2, m_2m_1, w_2w_2$$
$$\succ_{m_3}: w_1w_3, w_2w_3, m_3m_3 \qquad \succ_{w_3}: m_1m_1, m_2m_2, m_3m_3, m_1m_3, m_2m_3, w_3w_3$$

For all men, $\succ_m \in \mathcal{C}_m$. For all women, $\succ_w \in \mathcal{I}_w$. All ex ante stable matchings are summarized in Table C.2. Each can be blocked by a coalition of two men and two women.

Table C.2: All ex ante stable matchings in Example C.2 and blocking coalitions.

Matching	m_1	m_2	m_3	w_1	w_2	w_3	Blocking Coalition
μ^1	w_1w_2	w_2w_1	m_3m_3	m_1m_2	m_2m_1	w_3w_3	$\{m_2, m_3, w_2, w_3\}$
μ^2	m_1m_1	w_3w_2	w_2w_3	w_1w_1	m_3m_2	m_2m_3	$\{m_1, m_3, w_1, w_3\}$
μ^3	w_3w_1	m_2m_2	w_1w_3	m_3m_2	w_2w_2	m_1m_3	$\{m_1, m_2, w_1, w_2\}$
μ^4	m_1m_1	m_2m_2	m_3m_3	w_1w_1	w_2w_2	w_3w_3	$\{m_1, m_2, w_1, w_2\}$

Even when one side of the market has preferences with strong inertia, volatile matchings can be in the dynamic core.

Example C.3. Consider a market with two men and two women whose preferences are

$$\succ_{m_1}: w_1w_2, w_2w_2, w_1w_1, m_1m_1 \qquad \succ_{w_1}: m_1m_1, m_2m_2, m_1m_2, w_1w_1$$
$$\succ_{m_2}: w_2w_1, w_1w_1, w_2w_2, m_1m_1 \qquad \succ_{w_2}: m_2m_2, m_1m_1, m_2m_1, w_2w_2$$

For all men, $\succ_m \in \mathcal{C}_m^*$. For all women, $\succ_w \in \mathcal{I}_w \cap \bar{\mathcal{S}}_w$. All three dynamically stable matchings (Table C.3) are in the dynamic core.

Table C.3: All dynamically stable and core matchings in Example C.3.

Matching	m_1	m_2	w_1	w_2
μ^1	w_1w_2	w_2w_1	m_1m_2	m_2m_1
μ^2	w_2w_2	w_1w_1	m_2m_2	m_1m_1
μ^3	w_1w_1	w_2w_2	m_1m_1	m_2m_2

In our main analysis both sides of the market were ex ante symmetric. In contrast, Theorem C.1 accommodates an asymmetry concerning admissible preference domains. The type of asymmetry considered may apply in some applications. For example, in a school-assignment application, it may be appropriate to assume that the priority structure (schools' "preferences") satisfies the strong inertia requirement (Kennes et al., 2014a). Once enrolled at a school, the student can stay enrolled in future years without fear of being "bumped" by a new student. Students' preferences can satisfy the weaker condition accommodating a taste for variety. A consequence of this asymmetry is that a volatile matching may be a core outcome and preferred by some agents. In Example C.3, for instance, both men prefer the volatile core matching μ^1 to every other dynamically stable outcome.

D Additional Examples

Example D.1 (Kadam and Kotowski (2015))**.** This example illustrates that there may not exist an optimal dynamically stable matching for all agents on one side of the market. A stable matching is said to be optimal for the men if all men prefer their assignment in that matching to every other stable matching. A woman-optimal stable matching is defined analogously. In Example 4 there does not exist a woman-optimal stable matching. This example employs a more restrictive class of preferences.

There are three men and three women. Their preferences are:

$$\succ_{m_1}: w_2w_2, w_2w_1, w_2w_3, w_1w_1, w_3w_3, m_1m_1$$
$$\succ_{m_2}: w_3w_3, w_3w_2, w_3w_1, w_2w_2, w_1w_1, m_2m_2$$
$$\succ_{m_3}: w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3m_3$$

$$\succ_{w_1}: m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, m_3m_3$$
$$\succ_{w_2}: m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, w_2w_2, m_1m_1$$
$$\succ_{w_3}: m_1m_1, m_1m_3, m_3m_1, w_3m_1, m_2m_1, m_3m_3, w_3w_3, m_2m_2$$

All preferences satisfy the $\succ_i \in \bar{\mathcal{S}}_i$ condition. There are three dynamically stable matchings (Table D.1). m_1 and m_2 like their assigned plans in μ^3 the most. m_3 prefers his assignment under μ^1.

Table D.1: All dynamically stable matchings in Example D.1.

Matching	m_1	m_2	m_3	w_1	w_2	w_3
μ^1	w_1w_1	w_2w_2	w_3w_3	m_1m_1	m_2m_2	m_3m_3
μ^2	w_3w_3	w_1w_1	w_2w_2	m_2m_2	m_3m_3	m_1m_1
μ^3	w_2w_3	w_3w_1	w_1w_2	m_3m_2	m_1m_3	m_2m_1

This example also shows that dynamically stable matchings do not form a lattice under the "common-preference" partial order (cf. Knuth (1976), Roth (1985b), and Blair (1988)). Kadam and Kotowski (2015) identify sufficient conditions for the set of dynamically stable matchings to exhibit a lattice structure, albeit under an alternative ordering.

Example D.2. This example exhibits ex ante and dynamically stable matchings that are not Pareto optimal. Consider the following economy with two men and women whose preferences are

$$\succ_{m_1}: w_1w_2, w_2w_2, w_1w_1, m_1m_1 \qquad \succ_{w_1}: m_1m_2, m_2m_2, m_1m_1, w_1w_1$$
$$\succ_{m_2}: w_2w_1, w_1w_1, w_2w_2, m_2m_2 \qquad \succ_{w_2}: m_2m_1, m_1m_1, m_2m_2, w_2w_2$$

The agents' preferences satisfy sequential improvement complementarity. There are exactly two ex ante and dynamically stable matchings as summarized in Table D.2. The matching μ^1 Pareto-dominates μ^2. The P-DA and the P-DAA procedure identify the μ^2 matching.

Table D.2: All ex ante and dynamically stable matchings in Example D.2.

Matching	m_1	m_2	w_1	w_2
μ^1	w_1w_2	w_2w_1	m_1m_2	m_2m_1
μ^2	w_2w_2	w_1w_1	m_2m_2	m_1m_1

Example D.3. Consider a market with two men and women. Their preferences are:

$$\succ_{m_1}: w_1w_2, w_1m_1, w_2w_2, w_2m_1, m_1m_1 \qquad \succ_{w_1}: m_1m_1, m_1w_1, w_1w_1$$
$$\succ_{m_2}: w_2w_2, m_2m_2 \qquad \succ_{w_2}: m_1m_1, m_2m_1, m_2m_2, w_2w_2$$

The preferences of m_1 do not satisfy SIC(2). Otherwise, SIC is satisfied. This economy does not have a dynamically stable matching. There are five possible cases.

1. If $\mu(m_1) = w_1w_2$, then $\mu(w_1) = m_1w_1$ and $\mu(w_2) = m_2m_1$. However, this implies that $m_2m_2 \succ_{m_2} \mu(m_2)$.

2. If $\mu(m_1) = w_1m_1$, then $\mu(w_1) = m_1w_1$. Furthermore, $\mu(m_2) = w_2w_2$ and $\mu(w_2) = m_2m_2$. However, m_1 and w_2 can period-2 block μ.

3. If $\mu(m_1) = w_2w_2$, then $\mu(w_1) = w_1w_1$. However, m_1 and w_1 can period-1 block this matching.

4. If $\mu(m_1) = w_2m_1$, then $w_2w_2 \succ_{w_2} \mu(w_2)$.

5. If $\mu(m_1) = m_1m_1$, then $\mu(w_1) = w_1w_1$. However, m_1 and w_1 can period 1 block μ.

The preceding cases exhaust all individually rational matchings for m_1. Therefore, there does not exist a dynamically stable matching.

Example D.4. Consider a market with two men and women. Their preferences are:

$$\succ_{m_1}: m_1w_2, m_1w_1, w_1w_1, w_2w_2, m_1m_1 \qquad \succ_{w_1}: w_1m_1, m_1m_1, w_1w_1$$
$$\succ_{m_2}: w_2w_2, m_2m_2 \qquad \succ_{w_2}: m_2m_1, m_1m_1, m_2m_2, w_2m_1, w_2w_2$$

The preferences of m_1 do not satisfy SIC(3). Otherwise, SIC is satisfied. This economy does not have a dynamically stable matching. There are five possible cases.

1. If $\mu(m_1) = m_1w_2$, then $\mu(w_2) = w_2m_1$. However, m_2 and w_2 can period-1 block this matching.

2. If $\mu(m_1) = m_1w_1$, then $\mu(w_1) = w_1m_1$. It follows that $\mu(m_2) = w_2w_2$ and $\mu(w_2) = m_2m_2$. However, now m_1 and w_2 can period-2 block the resulting matching.

3. If $\mu(m_1) = w_1w_1$, then m_1 and w_1 can period-1 block this matching.

4. If $\mu(m_1) = w_2w_2$, then $\mu(w_1) = w_1w_1$. However, this implies m_1 and w_1 can period-1 block this matching.

5. If $\mu(m_1) = m_1m_1$, then m_1 will period-1 block with either w_1 or w_2.

The preceding cases exhaust all individually rational matchings for m_1. Therefore, there does not exist a dynamically stable matching.

Example D.5. This example shows that if agents' preferences satisfy sequential improvement complementarity, then every matching mechanism that identifies a dynamically stable outcome (if one exists) can be manipulated by at least one man and at least one woman when agents are unrestricted in the preferences that they report.

Consider the following market with three men and three women:

$$\succ_{m_1}: \boxed{w_1w_2}, \underline{w_2w_2}, m_1m_1$$
$$\succ_{m_2}: w_2w_2, \underline{w_1w_3}, \boxed{w_2w_3}, m_2m_2$$
$$\succ_{m_3}: w_2w_2, \boxed{\underline{w_3w_1}}, w_1w_1, m_3m_3$$

$$\succ_{w_1}: m_1m_1, \underline{m_2m_3}, \boxed{m_1m_3}, w_1w_1$$
$$\succ_{w_2}: \boxed{m_2m_1}, \underline{m_1m_1}, w_2w_2$$
$$\succ_{w_3}: m_1m_1, \boxed{\underline{m_3m_2}}, m_2m_2, w_3w_3$$

Table D.3: All dynamically stable matchings in Example D.5.

Matching	m_1	m_2	m_3	w_1	w_2	w_3
μ^1	w_1w_2	w_2w_3	w_3w_1	m_1m_3	m_2m_1	m_3m_2
μ^2	w_2w_2	w_1w_3	w_3w_1	m_2m_3	m_1m_1	m_3m_2
μ^3	w_2w_2	m_2m_2	m_3m_3	w_1w_1	m_1m_1	w_3w_3

There are three dynamically stable matchings as summarized in Table D.3. Matching μ^1 is boxed in the preference list. Matching μ^2 is underlined. Matching μ^3, which corresponds to the P-DAA matching, is not highlighted. Thus, there are three cases to consider.

1. Consider a matching mechanism that selects the matching μ^1 if all agents truthfully report their preferences.

   (a) If m_2 claims the preference profile

   $$\hat{\succ}_{m_2}: w_1w_3, w_3m_2, w_3w_3, m_2m_2$$

   and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with μ^2, which m_2 prefers.

   (b) If w_1 claims the preference profile

   $$\hat{\succ}_{w_1}: m_2m_3, m_1w_1, m_1m_1, w_1w_1$$

   and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with μ^2, which w_1 prefers.

2. Consider a matching mechanism that selects the matching μ^2 if all agents truthfully report their preferences.

   (a) If m_1 claims the preference profile

   $$\hat{\succ}_{m_1}: w_1w_2, w_1m_1, w_1w_1, w_2m_1, w_2w_2, w_3m_1, w_3w_3, m_1m_1$$

   and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with μ^1, which m_1 prefers.

(b) If w_2 claims the preference profile

$$\hat{\succ}_{w_2}: m_2m_1, m_1w_2, m_1m_1, m_2w_2, m_2m_2, m_3w_2, m_3m_3, w_2w_2$$

and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with μ^1, which w_2 prefers.

3. Consider a matching mechanism that selects the matching μ^3 if all agents truthfully report their preferences. In this case, any of the above manipulations benefit the responsible agent.

Every matching mechanism that selects a dynamically stable matching must select one of the above three matchings in the economy above. In every case there exists one man and one woman who can successfully manipulate the mechanism for their advantage by communicating a preference that does not satisfy sequential improvement complementarity.

E The Deferred Acceptance Algorithm

We often reference the deferred acceptance algorithm of Gale and Shapley (1962). Though well-known, we review this procedure below as it applies to a one-period market. Each man m (woman w) has a strict preference ranking, P_m (P_w), over potential partners in W_m (M_w). If iP_mj, then m strictly prefers i to j.

Definition E.1. The *(man-proposing) deferred acceptance algorithm* constructs a (one-period) matching μ as follows:

1. In round 1, each man proposes to his most preferred partner as defined by P_m. (If mP_mw for all $w \in W$, he does not make any proposals.) Given all received proposals, each woman engages her most preferred partner as defined by P_w and rejects the others. All proposals from unacceptable partners (i.e. ranked below w by P_w) are rejected.

2. More generally, in round t, each man whose proposal was rejected in the previous round proposes to his most preferred partner who has not yet rejected him. If all such partners are unacceptable, he does not make any proposals. Out of the set of new proposals and her current engagement (if any), each woman engages her most preferred partner and rejects the others. If all proposals are unacceptable, she rejects them all.

The above process stops once no further rejections occur. At that time all engaged pairs are matched and agents without a partner remain single (i.e. are matched to themselves).

The woman-proposing deferred acceptance algorithm is identical to the procedure described above with the roles of men and women reversed. The next example illustrates the algorithm's operation.

Example E.1. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. The agents' preferences are:

$$
\begin{array}{ll}
P_{m_1}: w_2, w_1, m_1 & \quad P_{w_1}: m_1, m_2, m_3, w_1 \\
P_{m_2}: w_1, w_3, m_2 & \quad P_{w_2}: m_3, m_1, w_2 \\
P_{m_3}: w_1, w_2, m_3 & \quad P_{w_3}: m_2, w_3
\end{array}
$$

That is, m_1 prefers w_2 to w_1. He prefers either to being single. w_3 is not acceptable.

Table E.1 summarizes the round-by-round operation of the man-proposing deferred acceptance algorithm. To read the table, in round 1, m_2 and m_3 propose to w_1. She engages

m_2, who is underlined, and m_3 is rejected. m_1 proposes to w_2 and is engaged. No one proposes to w_3. Eventually we arrive at the final matching:

$$\begin{array}{lll} \mu(m_1) = w_1 & \mu(m_2) = w_3 & \mu(m_3) = w_2 \\ \mu(w_1) = m_1 & \mu(w_2) = m_3 & \mu(w_3) = m_2 \end{array}.$$

Table E.1: Round-by-round operation of the deferred acceptance algorithm in Example E.1. Engaged partners are underlined.

	Proposals Received		
Round	w_1	w_2	w_3
1	$\underline{m_2}, m_3$	$\underline{m_1}$	-
2	$\underline{m_2}$	$m_1, \underline{m_3}$	-
3	$\underline{m_1}, m_2$	$\underline{m_3}$	-
4	$\underline{m_1}$	$\underline{m_3}$	$\underline{m_2}$

E.1 The Spot-Market Deferred Acceptance Procedure (Example 5)

Example 5 identifies a case where the spot-market deferred acceptance procedure generates an unstable outcome. Here we provide details concerning that market's operation.

Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. The agents' preferences are:

$$\begin{array}{ll} \succ_{m_1}: w_2w_2, w_1w_2, w_1w_1, \ldots & \succ_{w_1}: m_1m_1, m_2m_2, m_3m_3, m_1m_2, m_1m_3, \ldots \\ \succ_{m_2}: w_1w_1, w_3w_3, w_3w_1, \ldots & \succ_{w_2}: m_3m_3, m_1m_1, m_3m_1, \ldots \\ \succ_{m_3}: w_1w_1, w_2w_1, w_2w_2, \ldots & \succ_{w_3}: m_2m_2, \ldots \end{array}$$

Given $\succ_i$ we can define each agent's ex ante spot ranking:

$$\begin{array}{ll} P_{\succ_{m_1}}: w_2, w_1, \ldots & P_{\succ_{w_1}}: m_1, m_2, m_3, \ldots \\ P_{\succ_{m_2}}: w_1, w_3, \ldots & P_{\succ_{w_2}}: m_3, m_1, \ldots \\ P_{\succ_{m_3}}: w_1, w_2, \ldots & P_{\succ_{w_3}}: m_2, \ldots \end{array}$$

These same preferences appear in Example E.1. Constructing $\tilde{\mu}_1$ via the man-proposing deferred acceptance algorithm gives:

$$\begin{array}{lll} \tilde{\mu}_1(m_1) = w_1 & \tilde{\mu}_1(m_2) = w_3 & \tilde{\mu}_1(m_3) = w_2 \\ \tilde{\mu}_1(w_1) = m_1 & \tilde{\mu}_1(w_2) = m_3 & \tilde{\mu}_1(w_3) = m_2 \end{array}$$

At $\tilde{\mu}_1(\cdot)$, agents' conditional spot rankings are:

$$\begin{array}{ll} P^{w_1}_{\succ_{m_1}} : w_2, w_1, \ldots & P^{m_1}_{\succ_{w_1}} : m_1, m_2, m_3, \ldots \\ P^{w_3}_{\succ_{m_2}} : w_3, w_1, \ldots & P^{m_3}_{\succ_{w_2}} : m_3, m_1, \ldots \\ P^{w_2}_{\succ_{m_3}} : w_1, w_2, \ldots & P^{m_2}_{\succ_{w_3}} : m_2, \ldots \end{array}$$

Using the above spot rankings we can construct $\tilde{\mu}_2$ via the man-proposing deferred acceptance algorithm. In this case, the resulting period 2 assignment is:

$$\begin{array}{lll} \tilde{\mu}_2(m_1) = w_2 & \tilde{\mu}_2(m_2) = w_3 & \tilde{\mu}_2(m_3) = w_1 \\ \tilde{\mu}_2(w_1) = m_3 & \tilde{\mu}_2(w_2) = m_1 & \tilde{\mu}_2(w_3) = m_2 \end{array}$$

And the resulting multi-period matching is:

$$\begin{array}{lll} \tilde{\mu}(m_1) = w_1 w_2 & \tilde{\mu}(m_2) = w_3 w_3 & \tilde{\mu}(m_3) = w_2 w_1 \\ \tilde{\mu}(w_1) = m_1 m_3 & \tilde{\mu}(w_2) = m_3 m_1 & \tilde{\mu}(w_3) = m_2 m_2 \end{array}$$

This matching is neither ex ante nor dynamically stable. For example, m_1 and w_2 can period-1 block $\tilde{\mu}$ since $w_2 w_2 \succ_{m_1} \tilde{\mu}(m_1)$ and $m_1 m_1 \succ_{w_2} \tilde{\mu}(w_2)$.