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Sangram V. Kadam

Harvard University

Maciej H. Kotowski

Harvard Kennedy School

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# Time Horizons, Lattice Structures, and Welfare in Multi-period Matching Markets\*

Sangram V. Kadam<sup>†</sup>      Maciej H. Kotowski<sup>‡</sup>

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## Abstract

We analyze a  $T$ -period, bilateral matching economy without monetary transfers. Under natural restrictions on agents' preferences, which accommodate switching costs, status-quo bias, and other forms of inter-temporal complementarity, dynamically stable matchings exist. We propose an ordering of the set of dynamically stable matchings ensuring this set forms a lattice. We investigate the robustness of dynamically stable matchings with respect to the market's time horizon. We relate our analysis to market-design applications, including student-school assignment and labor markets.

Keywords: Two-sided Matching, Dynamic Matching, Stable Matching, Multi-period Matching, Lattice, Market-design

JEL: C78, D47, C71

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<sup>†</sup>Department of Economics, Harvard University, 1805 Cambridge Street, Cambridge MA 02138. E-mail: <svkadam@fas.harvard.edu>

<sup>‡</sup>(Corresponding Author) John F. Kennedy School of Government, Harvard University, 79 JFK Street, Cambridge MA 02138. E-mail: <maciej\_kotowski@hks.harvard.edu>

The passage of time is a key component of many social and economic interactions. Wedding anniversaries are celebrated—or forgotten. Employees are recognized for years of service. And students receive an education at a succession of institutions, from pre-school to high school to, possibly, graduate school. Sometimes persistence is held in high regard. At other times, change is eagerly anticipated.

Though time is an ingredient of many economic models, it is largely absent from typical studies of bilateral matching markets, as originally formulated by Gale and Shapley (1962). In such a market agents are partitioned into two groups—men and women, firms and workers, schools and students—and seek to match together to realize benefits. Crucially, each agent has preferences defined over potential partners and these preferences need not be aligned. In Gale and Shapley’s classic terminology, which we adopt solely for its simplicity, men and women sometimes (dis)agree in their evaluations of each other. Likewise, firms value a particular worker’s skills differently and his ranking of employers may be equally idiosyncratic. That “stable” matchings, where no agent or no pair can pursue a mutually-preferable arrangement to a proposed aggregate assignment, are possible is both surprising and profound.

Extending a matching market’s time horizon, forces one to confront several real-world complications. First, with the passage of time agents frequently change their partners.<sup>1</sup> The possibility of change introduces a degree of complexity not encountered in the single-period, one-shot case. The order of interactions matters and inter-temporal trade-offs assume a prominent role. Second, practical decision making over a time horizon is difficult. Complexity and psychology mix to promote habit formation and path-dependence into significant behavioral features. Unlike theory, practice often conflicts with Samuelson’s (1937) influential discounted-utility framework. Finally, when time horizons are long, agents’ commitment ability is imperfect and dynamic incentives matter.

On a normative front, many classic applications of matching theory have a multi-period structure, though it has not been traditionally emphasized or exploited. For example, the assignment of students to high schools has emerged as a celebrated application of matching theory (Abdulkadiroğlu and Sönmez, 2003). Traditional models of this problem have

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<sup>1</sup>Change occurs in all celebrated applications of matching theory. Only 54.4 percent of couples married between 1975 and 1979 celebrated their 25th anniversary (Kreider and Ellis, 2011). Similarly, the U.S. Bureau of Labor Statistics (2012) notes that the average person born between 1957 and 1964 held 11.3 jobs between the ages of 18 and 46. Non-employment accounted for 22 percent of weeks during this period. Finally, the U.S. public high school adjusted cohort graduation rate for school year 2011–12 was 80 percent (Stetser and Stillwell, 2014). Thus, twenty percent of students “re-match” with some alternative(s) to a public high school education. This may range from dropping out entirely to pursuing an alternative credential.

been static, one-period affairs. Of course, students attend high school for (typically) four years and, therefore, this is a multi-period assignment problem. A very coarse conflation of the successive school years yields the typical one-shot model; however, an unbundling of successive years, semesters, or quarters yields a rich family of multi-period alternatives. In fact, assignment length and re-matching frequency are in principle design variables open to refinement.<sup>2</sup> Going beyond one period is necessary, first, to understand and, second, to leverage inter-temporal tradeoffs in matching problems.

Though we are mindful of applications, the primary purpose of our study is to examine the positive theory of  $T$ -period, bilateral matching economies where agents' preferences exhibit non-trivial, inter-temporal complementarities. We focus on cases where agents are limited in their commitment ability and must be continually incentivized to continue with a proposed assignment plan. Formally, our analysis builds upon Kadam and Kotowski (2015), who analyze a multi-period matching economy focusing on “dynamically stable” matchings. Whereas Kadam and Kotowski (2015) primarily address applications of their baseline model, the analysis below focuses on the structural and welfare properties of dynamically stable outcomes. Thus, we provide a deeper and complementary characterization of stable outcomes.

Along multiple dimensions, our analysis and conclusions differ from those of Damiano and Lam (2005), Kurino (2009), or Pereyra (2013) where agents' preferences are “time separable.” In a multi-period matching market, such a condition is a substantive restriction since status-quo bias, switching costs, and other forms of preference inertia are common. To accommodate these features, we first generalize a class of preferences studied by Kadam and Kotowski (2015). Such preferences merge a ranking of potential partners with a bias toward more persistent assignments, which we call inertia. In the two-period case, they satisfy the “rankability” condition of Kennes et al. (2014a), which is among the few other studies incorporating inter-temporal preference complementarities.<sup>3</sup> Contrary to connotation, inertia need not reinforce a matching's stability. Interim preference reversals render the matching problem more nuanced than a sequence of independent, single-period markets.

Despite a bias toward persistent matchings, dynamic stability and volatility in assignments are not only compatible, but surprisingly common. Stable matchings may involve change at the agent level at every opportunity. At times, stability may necessitate periods of

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<sup>2</sup>For example, educational programs may be able to alleviate capacity concerns by building-in re-matchings or rotations into initial assignments.

<sup>3</sup>Kennes et al. (2014a) define rankability only for the two-period case. Therefore, our analysis may be interpreted as a generalization of their condition to a  $T$ -period setting. However, our construction and motivation for such preferences is closer to the exposition of Kadam and Kotowski (2015).

sacrifice where agents pair temporarily with (extremely) low-quality partners in anticipation of a better pairing later on. Similarly, agents may experience isolated periods of being unmatched. In a labor market application, we often call such cases job-hopping,<sup>4</sup> internships, and unemployment, respectively. Our model accommodates all of them and can form a basis for further study of labor market dynamics and career planning. We show how many of these outcomes require a long time horizon to be stable market arrangements. Importantly, they need not follow from an intrinsic “preference for variety” and may emerge as a compromise among otherwise conflicting interests.

After reviewing the literature in Section 1 and introducing our model in Section 2, we address two questions hitherto unexplored in multi-period bilateral matching economies. Both are salient for applications as they address welfare and robustness. First, in Section 3, we investigate the lattice structure of the set of dynamically stable matchings.<sup>5</sup> Such a structure provides economic insight concerning the way markets mediate conflicting interests, facilitates welfare comparisons, and frequently simplifies formal arguments. Though characterized in static applications,<sup>6</sup> the nature of the lattice of stable matchings in dynamic markets has not been explored. Our analysis yields subtle conclusions and qualifications. First, we identify cases where dynamically stable matchings are not Pareto optimal. Moreover, we show that man- or woman-optimal stable matchings do not always exist.<sup>7</sup> Man- and woman-optimal matchings exist in a wide range of one-period models (Gale and Shapley, 1962; Roth, 1984, 1985b). Therefore, our environment is a substantive departure from a one-period setting and our assumptions hardly “reduce” our setting to a static environment. As a counterpoint to these intriguing though negative observations, we propose a refinement on agents’ preferences and a new ordering of the set of dynamically stable matchings ensuring that this set is a lattice. The new ordering maintains a link to agents’ preferences, thereby accommodating comparative welfare analysis.

While the set of stable matchings offers a rich collection of welfare implications, this set’s robustness with respect to slight changes in the environment remains unexplored in a dynamic setting. As our analysis emphasizes the multi-period nature of agents’ interactions,

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<sup>4</sup>“Job hopping” refers to the rapid movement of workers between firms. Fallick et al. (2006) document this phenomenon among technology-sector workers in Silicon Valley.

<sup>5</sup>A lattice is a partially-ordered set where any two elements have a unique supremum and a unique infimum.

<sup>6</sup>Knuth (1976), Blair (1984, 1988), Roth (1984, 1985b), and Alkan (2001, 2002), study the lattice structure of the stable set in an array of matching models. See also Sotomayor (1999).

<sup>7</sup>A stable matching is man-optimal if each man prefers his assignment in that matching to his assignment in all other stable matchings. A woman-optimal stable matching is defined analogously.

it is important to understand how the set of stable matchings may change with the market’s time horizon, the parameter  $T$  in our notation. This is the second focus of our analysis and it is significant for two reasons. First, in many positive analyses the selection and interpretation of “ $T$ ” is at the analyst’s discretion. Thus, a hint of robustness along this dimension is desirable. In Section 4 we show that many conclusions are robust to changes in  $T$ . Namely, intuitive projections and embeddings of stable outcomes are possible as  $T$  changes. Such conclusions are not immediate since, for instance, adjusting a market’s time horizon increases/decreases the number of blocking opportunities. Second, in normative analyses, the market’s time horizon provides a new design variable open to refinement and fine-tuning. Understanding the consequences of more or fewer re-matching opportunities can help guide design.

We view our focus on the above questions as complementary and providing a preliminary tool-kit for future applications. Both are relevant for analyses of markets where the time horizon is long but agents’ commitment ability is limited. As we discuss before the conclusion, specific applications can include studies of student-school or job-worker assignments.

The Appendix presents proofs not included in the main text.

## 1 Related Literature

Dynamic models occupy a nook within the expansive literature following Gale and Shapley (1962). Some papers, of which Kurino (2014) is a recent example, consider one-sided problems. In contrast, we consider a two-sided market where agents on both sides of the market, men and women in our nomenclature, have preferences over their partner’s identity. Specific applications have motivated previous studies in this vein. For example, Kennes et al. (2014a) study the assignment of children to Danish daycares while Dur (2012) and Pereyra (2013) consider school-choice applications. Though these models incorporate features that are absent from our analysis, our model is not a special case of any of them.<sup>8</sup>

As our model generalizes Gale and Shapley’s (1962) original analysis, we maintain their pairwise focus in our preferred stability concept, which we term dynamic stability. Admittedly there are multiple plausible definitions of stability for dynamic matching economies. Some authors, such as Damiano and Lam (2005) and Kurino (2009), favor coalition-based

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<sup>8</sup>Our model is symmetric. Hence, even in the two-period case it is not a special case of the analysis of Kennes et al. (2014a), who maintain an asymmetry between the market’s two sides. Kadam and Kotowski (2015) show that a stable matching in the sense of Kennes et al. (2014a) may not satisfy the stability conditions that we employ.

definitions of stability with additional credibility qualifications. Independently, Doval (2014) also examines “dynamically stable” matchings in her model, which differs from our usage of the term. As elaborated upon by Kadam and Kotowski (2015), the pairwise notions of stability used below are intuitive generalizations of well-known concepts and they provide an accessible benchmark for further analysis.

A link exists between a dynamic, one-to-one matching market and a static, many-to-many matching market. Every agent is matched to many partners, though in succession. Our analysis is not a special case of the most general treatments of many-to-many matching markets lodged in matching with contracts framework (Hatfield and Milgrom, 2005). Notably, our model does not satisfy the substitutability condition stressed by Hatfield and Kominers (2012).

While dynamics have not received much attention in the literature on two-sided matching following Gale and Shapley (1962), they are a central pillar of alternative examinations of bilateral markets. For example, the fact that workers shuffle between jobs has been formalized in the literature on matching within the search-theoretic paradigm. In the work of Mortensen and Pissarides (1994), to cite but one well-known example, workers move between employment and unemployment; job opportunities are created and destroyed. Micro-level change and churn are features of an economy’s steady state. We identify comparably volatile stable outcomes in our setting and we hope our analysis serves as a useful step toward integrating insights from both “matching” literatures.<sup>9</sup>

## 2 Model

Let  $M$  and  $W$  be finite, disjoint sets of agents—men and women, respectively—who interact over  $T$  periods. In every period, each man (woman) can be matched with one woman (man) or remain single. Following convention, an unmatched agent is said to be “matched to him/herself.” Thus, the set of potential partners for  $m \in M$  is  $W_m := W \cup \{m\}$ .  $M_w := M \cup \{w\}$  is the set of potential partners for  $w \in W$ . We use  $m$ ’s and  $w$ ’s to denote specific men and women when a distinction is helpful. Otherwise,  $i, j, k, l$  are generic agents.

*Remark 1.* For brevity, we define some concepts only from the perspective of a typical man. Our model is symmetric and all definitions apply to women with obvious notational changes.

Over a lifetime each agent encounters a sequence of partners, called a partnership plan. We denote a plan by  $x = (x_1, x_2, \dots, x_T)$  where  $x_t$  is the assigned partner in period  $t$ . When

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<sup>9</sup>The links between these literatures was recognized by Crawford and Knoer (1981, p. 437–8).

confusion is unlikely, we write  $x = x_1x_2 \cdots x_T$  with truncations or subsets of  $x$  denoted as follows:

$$\begin{aligned} x_{\leq t} &= x_1x_2 \cdots x_t & x_{\geq t} &= x_tx_{t+1} \cdots x_T & x_{[t,t']} &= x_t \cdots x_{t'} \\ x_{< t} &= x_1x_2 \cdots x_{t-1} & x_{> t} &= x_{t+1}x_{t+2} \cdots x_T & x_{(t,t')} &= x_{t+1} \cdots x_{t'-1} \end{aligned}$$

Often we combine and remix the above expressions, i.e.  $x = (x_{<t}, x_{[t,t']}, x_{>t})$ .<sup>10</sup> A constant plan is written as  $\bar{i} := (i, i, \dots, i)$ .

Each agent  $i$  has a strict, rational preference  $\succ_i$  defined over feasible partnership plans. If  $i$  prefers plan  $x$  to plan  $y$ , we write  $x \succ_i y$ . If  $x \succ_i y$  or  $x = y$ , then  $x \succeq_i y$ . We occasionally summarize  $\succ_i$  by listing plans in preferred order, i.e.  $\succ_i: x, y, z, \dots$

A *matching*  $\mu: M \cup W \rightarrow (M \cup W)^T$  assigns a partnership plan to each agent. It is a function comprising of  $T$  one-period matchings.

**Definition 1.**  $\mu_t: M \cup W \rightarrow M \cup W$  is a *one-period matching* (for period  $t$ ) if (i) for all  $m \in M$ ,  $\mu_t(m) \in W_m$ ; (ii) for all  $w \in W$ ,  $\mu_t(w) \in M_w$ ; and, (iii) for all  $i$ ,  $\mu_t(i) = j \implies \mu_t(j) = i$ .

Thus,  $\mu(i) = (\mu_1(i), \dots, \mu_T(i))$  is  $i$ 's plan under the matching  $\mu$ .

A matching is stable if it cannot be blocked by any agent or pair.

**Definition 2.** Agent  $i$  can *period- $t$  block* the matching  $\mu$  if  $(\mu_{<t}(i), \bar{i}_{\geq t}) \succ_i \mu(i)$ .

At period  $t$ ,  $(\mu_{<t}(i), \bar{i}_{\geq t})$  is the best outcome that agent  $i$  can guarantee himself independent of others' behavior or of the market's future development. If  $i$  cannot period- $t$  block  $\mu$  for all  $t$ , then his assignment is individually rational.

A pair can block a matching in period  $t$  if they can form a (continuation) partnership plan among themselves for periods  $t, t+1, t+2, \dots$ , that they both prefer given the elapsed history. Such a plan involves a sequence of arrangements only among the blocking pair.

**Definition 3.**  $\mu_t^{\{m,w\}}: \{m, w\} \rightarrow \{m, w\}$  is a *one-period matching* (for period  $t$ ) among  $\{m, w\}$  if (i)  $\mu_t^{\{m,w\}}(m) = w$  and  $\mu_t^{\{m,w\}}(w) = m$ ; or, (ii)  $\mu_t^{\{m,w\}}(m) = m$  and  $\mu_t^{\{m,w\}}(w) = w$ .

**Definition 4.**  $\{m, w\}$  can *period- $t$  block* the matching  $\mu$  if there exists a matching among  $\{m, w\}$ ,  $\mu^{\{m,w\}} = (\mu_1^{\{m,w\}}, \dots, \mu_T^{\{m,w\}})$ , such that  $(\mu_{<t}(i), \mu_{\geq t}^{\{m,w\}}(i)) \succ_i \mu(i)$  for all  $i \in \{m, w\}$ .

We call the matching  $\mu$  *dynamically stable* if it cannot be period- $t$  blocked by any agent or by any pair for all  $t$ . Dynamic stability is a natural and succinct multi-period generalization

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<sup>10</sup>By convention,  $x_{<1} = \emptyset$  and  $x_{>T} = \emptyset$ .



of Gale and Shapley’s original stability notion. Kadam and Kotowski (2015) discuss the solution concept at length and provide additional context for its formulation. Here, we comment on two of its features. First, we noted above that other studies favor coalition-based definitions of stability. Ours is a pairwise notion, though its coalition-based extension is straightforward and leads to a definition of the core.<sup>11</sup> Conditions ensuring the core’s non-emptiness are more restrictive than those favored in our analysis to follow, prompting our preference for the pairwise form (Kadam and Kotowski, 2015).

Second, dynamic stability imposes chronological restrictions on admissible blocking actions. Specifically, a blocking action initiated in period  $t$  carries consequences for all  $t' > t$ . Through such cross-period links, dynamic stability differs from a mere collection of ephemeral and independent one-period blocking actions. As an example,  $m$  and  $w$  cannot block a matching with a one-period partnership and then return to their planned assignment *as if* nothing ever happened.<sup>12</sup> Contemporaneously with a temporary deviation, for instance, the wider market might evolve in ways unexpected, or possibly unknown, to  $m$  and  $w$  given their period- $t$  action. The agents’ erstwhile partners may identify new partners and, since preferences need not be time-separable, a cascade of further re-matchings may ensue. Via the definition of period- $t$  blocking, dynamic stability posits that agents assess departures from a plan conjecturing a worst-case future development of the market as a whole. Acknowledging the situation’s complexity, particularly if  $T$  is large,  $m$  and  $w$  anticipate that they will be resigned to matching opportunities only among themselves—as the worst case—should they deviate from the prevailing matching. Though a conservative benchmark, the robust decision criterion implicit in the definition of dynamic stability contributes to its adaptability, extensibility, and portability.

When  $T = 1$  the above model reduces to Gale and Shapley’s (1962) matching market and their deferred acceptance algorithm identifies a stable matching.<sup>13</sup> Once  $T \geq 2$ , a dynamically stable matching may not exist and a restriction on agents’ preferences is necessary to ensure existence.<sup>14</sup> In many-to-one or many-to-many matching economies, for example, the well-known substitutability restriction steps in to play a similar role (Kelso and Craw-

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<sup>11</sup>It is sufficient to replace  $\{m, w\}$  in Definitions 3 and 4 with a coalition  $C \subset M \cup W$ .

<sup>12</sup>Damiano and Lam (2005) propose the concept of strict self-sustaining stability (“ $S^4$ ”) which allows agents to return to an original plan following a temporary deviation. They acknowledge that it presumes a non-deviating agent accepts a deviator’s return, which may not always be a desirable assumption.

<sup>13</sup>We rely on Gale and Shapley’s deferred acceptance algorithm at several points in our analysis. Roth (2008) summarizes the algorithm’s properties and wide application.

<sup>14</sup>To show non-existence, Kadam and Kotowski (2015) propose the following example. The man’s preferences are  $\succ_m : wm, ww, mm, \dots$  and the woman’s preferences are  $\succ_w : mm, ww, \dots$

ford, 1982; Roth, 1984). Given our setting’s dynamic nature, restrictions that nevertheless allow common forms of inter-temporal complementarities are desirable. Kadam and Kotowski (2015) identify a weak sufficient condition—*sequential improvement complementarity* (SIC)—that ensures the existence of a dynamically stable matching. When preferences satisfy SIC, a multi-period generalization of the deferred acceptance algorithm, the plan deferred acceptance procedure with adjustment (P-DAA), identifies a dynamically stable outcome.

Though SIC allows for a general model, it proves difficult to draw sharp positive conclusions without additional structure on agent’s preferences. Therefore, in the analysis below we further restrict attention to a class of preferences that *exhibit inertia relative to a spot ranking*. Such preferences merge a consistent ranking of potential partners and a bias for more-persistent plans. This latter element introduces complementarities in preferences *across* periods thereby leading to, speaking loosely, preference reversals. Kadam and Kotowski (2015) analyze such preferences only in the two-period context and show how they can be employed in applications of interest, including multi-period matching models with monetary transfers or incomplete information. Below we develop a new  $T$ -period generalization of this class of preferences using a novel preorder of partnership plans based on their relative persistence. As we explain, this class affords adequate tractability for analysis while remaining sufficiently rich for applications.

## 2.1 Preferences: Spot Rankings and Inertia

Imagine an agent holds a strict ranking of potential partners abstracting from all dynamic considerations. We call such a ranking a *spot ranking* and we denote it by  $P_i$ . If  $j$  ranks above  $k$ , we write  $jP_ik$ . If  $jP_ik$  or  $j = k$ , then  $jR_ik$ . The following definition proposes a link between  $P_i$  and the preference  $\succ_i$ .

**Definition 5.** The preference  $\succ_i$  *reflects*  $P_i$  if for all  $x$  and  $t$ ,

$$jP_ik \iff (x_1, \dots, x_{t-1}, j, x_{t+1}, \dots, x_T) \succ_i (x_1, \dots, x_{t-1}, k, x_{t+1}, \dots, x_T). \quad (1)$$

Let  $\mathcal{S}_i$  be the set of preferences for agent  $i$  that reflect a spot ranking.

When  $\succ_i \in \mathcal{S}_i$ ,  $i$  prefers the plan with the higher-ranked partner if two plans differ only in their assignment for one period. Definition 5 is similar to Roth’s (1985a) commonly-encountered “responsiveness” condition. Beyond holding the set of other partners fixed, our definition also requires the sequence of other partners on the righthand-side of (1) to be

fixed.<sup>15</sup> Of course, many distinct preferences may reflect the same spot ranking.

Though  $\mathcal{S}_i$  is an appealing class of preferences, it precludes many plausible situations. For example, if  $kP_i j$  and  $\succ_i \in \mathcal{S}_i$ , then agent  $i$  must believe that  $jkj \succ_i jjj$ . However, status-quo bias due to explicit or implicit switching costs may tilt  $i$ 's preference toward the  $jjj$  plan in lieu of plans incorporating many changes (Samuelson and Zeckhauser, 1988). In practice, plans economizing on switching may be held in relative favor even if the assignments are intrinsically less desirable in some time-independent sense.

To address the associated nuances, we first introduce a vocabulary describing partnership plan persistence and variability. The plan  $x = x_1 \cdots x_T$  is *maximally persistent* if  $x_t = x_{t'}$  for all  $t$  and  $t'$ . Otherwise it is *volatile*. It is *maximally volatile* if  $x_t \neq x_{t+1}$  for all  $t$ . Finally, the following comparison of plans' relative persistence will prove crucial.

**Definition 6.** The plan  $x$  is *more persistent than* plan  $y$ , denoted as  $x \supseteq y$ , if for all  $t \leq t'$ ,  $y_t = \cdots = y_{t'} \implies x_t = \cdots = x_{t'}$ .

Definition 6 defines a preorder of partnership plans. It is reflexive ( $x \supseteq x$ ) and transitive ( $x \supseteq y, y \supseteq z \implies x \supseteq z$ ), but not anti-symmetric. If  $x \supseteq y, y \supseteq x$ , but  $x \neq y$ , then  $x$  is *equally persistent to*  $y$  and we write  $x \bowtie y$ .  $x \triangleright y$  when  $x \supseteq y$  and  $y \not\supseteq x$ . Finally, we write  $x \parallel y$  if  $x$  and  $y$  are not  $\supseteq$ -comparable. To illustrate,  $ijj \triangleright ijk, iij \bowtie jjk$ , and  $ijj \parallel ijj$ .

To model a bias toward more persistent plans, we allow such plans to rise in an agent's preference ranking relative to a benchmark preference relation.

**Definition 7.** The preference  $\succ_i$  *exhibits inertia relative to*  $\succ'_i$  if

1.  $x \succ'_i y$  and  $x \supseteq y \implies x \succ_i y$ ; and,
2. for all  $x$  and  $y$  such that  $x \parallel y, x \succ'_i y \iff x \succ_i y$ .

Let  $\Upsilon(\succ'_i)$  be the set of preferences that exhibit inertia relative to  $\succ'_i$ .

Condition 2 in Definition 7 says that the relative rankings of non-comparable plans does not change vis-à-vis the  $\succ'_i$  baseline. A similar conclusion applies to equally-persistent plans. As with all omitted proofs, the following lemma's is proved in the Appendix.

**Lemma 1.** *If  $\succ_i \in \Upsilon(\succ'_i)$  and  $x \bowtie y$ , then  $x \succ'_i y \iff x \succ_i y$ .*

The following example illustrates the preceding definition and lemma.

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<sup>15</sup>Thus, if  $\succ_i \in \mathcal{S}_i$  and  $jP_i k$ , then  $hlj \succ_i hlk$ . However, it need not follow that  $hlj \succ_i lhk$ .

**Example 1.** Suppose  $W = \{w_1, w_2\}$  and consider the following preferences for  $m \in M$ :

$$\begin{aligned}\succ_m &: \underline{w_2w_2w_2}, w_2w_1w_1, \boxed{w_1w_2w_2}, w_1w_2w_1, \dots \\ \succ'_m &: w_2w_1w_1, w_1w_2w_1, \underline{w_2w_2w_2}, \boxed{w_1w_2w_2}, \dots\end{aligned}$$

$\succ_m$  exhibits inertia relative to  $\succ'_m$ . Here, the plans  $\underline{w_2w_2w_2}$  and  $\boxed{w_1w_2w_2}$  advanced in rank. Since  $\boxed{w_1w_2w_2} \bowtie w_2w_1w_1$ , these plans' relative ranking under  $\succ_m$  and  $\succ'_m$  is the same.

## 2.2 Stable Matchings

Formally, and in line with the above exposition, the ideas of a spot ranking and preference inertia are independent. However, together they combine to define a tractable family of preferences useful for analysis. Specifically, given  $\mathcal{S}_i$  we can define the set

$$\bar{\mathcal{S}}_i := \bigcup_{\succ'_i \in \mathcal{S}_i} \Upsilon(\succ'_i).$$

As shown by Theorem 1 below, when  $\succ_i \in \bar{\mathcal{S}}_i$  for all  $i$ , a dynamically stable matching exists. Theorem 1's proof draws on a technical lemma that we prove in the Appendix.

**Lemma 2.** *Suppose the matching  $\mu$  is dynamically individually rational for  $m \in M$  and  $w \in W$ . Suppose this couple can period- $t$  block  $\mu$ . If  $\succ_m \in \bar{\mathcal{S}}_m$  and  $\succ_w \in \bar{\mathcal{S}}_w$ , then  $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \mu(m)$  and  $(\mu_{<t}(w), \bar{m}_{\geq t}) \succ_w \mu(w)$ .*

**Theorem 1.** *If  $\succ_i \in \bar{\mathcal{S}}_i$  for all  $i$ , there exists a dynamically stable matching.*

*Proof.* We construct a dynamically stable matching using the (*man-proposing*) *ex ante deferred acceptance* (E-DA) procedure. Kadam and Kotowski (2015) study the same procedure in a two-period model. To specify this procedure, we first define the *ex ante spot ranking induced by  $\succ_i$* ,<sup>16</sup> denoted  $P_{\succ_i}$ , as  $jP_{\succ_i}k \iff \bar{j} \succ_i \bar{k}$ . It is straightforward to show that if  $\succ_i \in \Upsilon(\succ'_i)$ , then  $P_{\succ_i} = P_{\succ'_i}$ .

The E-DA procedure defines the matching  $\mu^*$  as follows: For each  $t$ ,  $\mu_t^*$  is the one-period matching identified by Gale and Shapley's (man-proposing) deferred acceptance algorithm when each agent  $i$  makes/accepts proposals according to his/her *ex ante spot ranking induced by  $\succ_i$* ,  $P_{\succ_i}$ . Below, we show that  $\mu^*$  is dynamically stable.

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<sup>16</sup>Kennes et al. (2014a) introduce a similar concept that they call the isolated preference relation. They use it to define their DA-IP mechanism. The DA-IP mechanism may generate dynamically unstable outcomes when  $\succ_i \in \bar{\mathcal{S}}_i$  for all  $i$  (Kadam and Kotowski, 2015).

First, we verify that  $\mu^*$  cannot be period- $t$  blocked by  $i$  alone. Suppose the contrary. By Lemma A.2 agent  $i$  can period-1 block  $\mu^*$ , i.e.  $\bar{i} \succ_i \mu^*(i)$ . But this implies  $iP_{\succ_i} \mu_i^*(i)$ . Thus, the one-period deferred acceptance algorithm assigned  $i$  to a partner whom he like less than being single—a contradiction.

It remains to verify that no pair of agents can period- $t$  block  $\mu^*$ . Again we argue by contradiction. Suppose  $m$  and  $w$  can period- $t$  block  $\mu^*$ . By Lemma 2, this implies  $\tilde{\mu}(m) = (\mu_{<t}^*(m), \bar{w}_{\geq t}) \succ_m \mu^*(m)$  and  $\tilde{\mu}(w) = (\mu_{<t}^*(w), \bar{m}_{\geq t}) \succ_w \mu^*(w)$ . Since  $\succ_m \in \bar{\mathcal{S}}_m$ ,  $\succ_m \in \Upsilon(\succ'_m)$  for some  $\succ'_m \in \mathcal{S}_m$ . As  $\mu^*(m) \supseteq \tilde{\mu}(m)$  and  $\tilde{\mu}(m) \succ_m \mu^*(m)$ , it follows that  $\tilde{\mu}(m) \succ'_m \mu^*(m)$ . This implies  $wP_{\succ'_m} \mu_i^*(m)$  and thus,  $wP_{\succ_m} \mu_i^*(m)$ . Likewise,  $mP_{\succ_w} \mu_i^*(w)$ . However, this implies  $m$  and  $w$  can block the one-period matching identified by the deferred acceptance algorithm when agents' preferences are given by  $P_{\succ_i}$ . This is a contradiction as the deferred acceptance algorithm identifies a stable one-period matching. As  $\mu^*$  cannot be period- $t$  blocked by any agent or by any pair, it is dynamically stable.  $\square$

As noted above, the E-DA procedure was initially studied in a two-period setting. The procedure's continued effectiveness suggests that our class of preferences has maintained the stability-assuring characteristics from the two-period case. It is also reassuring that a simple procedure, with well-established roots, leads to at least one stable matching in an otherwise complex setting. Nevertheless, our model yields a rich collection of outcomes beyond the persistent matching identified by the E-DA, as illustrated by the following example.

**Example 2.** Suppose  $M = \{m_1, m_2\}$ ,  $W = \{w_1, w_2\}$ , and  $T = 3$ . Table 1 summarizes agents' preferences, which satisfy the  $\succ_i \in \bar{\mathcal{S}}_i$  restriction. Table 2 lists all dynamically stable matchings in this economy. To read this table,  $\mu^1(m_1) = w_1 w_1 w_1$ , and so on. The man- and woman-proposing variants of the E-DA procedure identify the  $\mu^1$  matching.

To conclude this section, we make four observations motivated by Example 2. First, maximally-volatile plans can be dynamically stable, even when agents' preferences exhibit inertia. In fact, agents may prefer volatile outcomes among all dynamically stable matchings. For instance,  $w_1$  and  $w_2$  prefer  $\mu^2$  among the stable set. This observation may be surprising as preference inertia seemingly nudges agents to prefer more persistent plans.

Second, a stable matching may incorporate a period of sacrifice. In Example 2,  $\mu^2$  and  $\mu^3$  assign both men to a partner whom they rank *below* single-hood in an ex ante sense:  $m_1 P_{\succ_{m_1}} w_2$  and  $m_2 P_{\succ_{m_2}} w_1$ . Both men are rewarded with a highly-ranked partner in the final period. Thus, with a long enough time horizon costly actions can be adequately incentivized

Table 1: Agent’s preferences in Example 2.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{w_1}$	$\succ_{w_2}$
$w_1w_1w_1$	$w_2w_2w_2$	$m_2m_2m_2$	$m_1m_1m_1$
$w_1m_1w_1$	$w_2m_2w_2$	$m_1m_2m_2$	$m_2m_1m_1$
$w_1w_2w_1$	$w_2w_1w_2$	$m_2m_2m_1$	$m_1m_1m_2$
$m_1m_1w_1$	$m_2m_2w_2$	$m_1m_2m_1$	$m_2m_1m_2$
$w_1m_1m_1$	$w_2m_2m_2$	$w_1m_2m_2$	$w_2m_1m_1$
$m_1w_2w_1$	$m_2w_1w_2$	$m_2m_2w_1$	$m_1m_1w_2$
$m_1m_1m_1$	$m_2m_2m_2$	$w_1m_2m_1$	$w_2m_1m_2$
		$m_1m_1m_1$	$m_2m_2m_2$
		$w_1w_1w_1$	$w_2w_2w_2$

Table 2: All dynamically stable matchings in Example 2.

Matching	$m_1$	$m_2$	$w_1$	$w_2$
$\mu^1$	$w_1w_1w_1$	$w_2w_2w_2$	$m_1m_1m_1$	$m_2m_2m_2$
$\mu^2$	$w_1w_2w_1$	$w_2w_1w_2$	$m_1m_2m_1$	$m_2m_1m_2$
$\mu^3$	$m_1w_2w_1$	$m_2w_1w_2$	$w_1m_2m_1$	$w_2m_1m_2$

as part of a stable outcome. Thus, entirely “per-period” characterizations of individual rationality may fail to capture the importance of these inter-temporal linkages.

Third, stable outcomes may include periods of temporary single-hood. The matching  $\mu^3$  in Example 2 has the surprising property that all agents are unmatched in period 1 but matched in later periods. This outcome has no analogue when  $T = 1$  and is not possible when  $T = 2$  (Lemma A.3). It suggests a new qualification of Roth’s (1986) rural hospital theorem. When the time horizon is short ( $T \leq 2$ ), Roth’s (1986) conclusion applies, but the relationship breaks down when  $T \geq 3$ .<sup>17</sup> Example 2 shows that if an agent is unmatched in some period in one dynamically stable matching, s/he may be matched in that period in some other dynamically stable matching. Such an outcome would not be possible if the economy was simply a sequence of independent, one-period matching interactions.

It is instructive to relate the preceding observations to contemporary labor markets. With this interpretation, the  $\mu^3$  matching exhibits a simultaneous bout of temporary unemployment when all agents are unmatched. One interpretation for this outcome is that of a

<sup>17</sup>This qualification presumes that  $\succ_i \in \mathcal{S}_i$  for all  $i$ .

business cycle due to market mis-coordination. Furthermore, institutional features implicit in our definition of blocking serve to reinforce this outcome. For example,  $m_1$  and  $w_1$  would prefer to be partnered in period 1, but they cannot period-1 block  $\mu^3$ .  $w_1$  is not keen on the long-term relationship with  $m_1$  that period-1 blocking implies (Lemma 2). Though stylized, the implicit persistence of blocking actions captures some common labor-market rigidities.

Finally, dynamically stable matchings may not be Pareto optimal.<sup>18</sup> When  $T = 1$ , stable matchings are Pareto optimal (Gale and Shapley, 1962). Example 2 shows that this conclusion is not true at a longer time horizon.<sup>19</sup> For all  $i$ ,  $\mu^2(i) \succ_i \mu^3(i)$ . Several peculiarities of this example—such as the temporary single-hood in  $\mu^3$ —suggest that non-Pareto optimal dynamically stable matchings occur under unusual circumstances. Such outcomes can occur in more ordinary cases as well. We prove the following theorem with Example 5, which will be presented in the next section.

**Theorem 2.** *If  $T \geq 2$  and  $\succ_i \in \bar{S}_i$  for all  $i$ , a dynamically stable matching may be Pareto-dominated by another dynamically stable matching.*

### 3 Welfare and Conflicting Interests

Conventional wisdom suggests that agents on different sides of a market have opposing interests. A seller prefers a price hike while a buyer desires a discount. Similarly, agents on the same side of the market compete for lucrative trading opportunities. A competitor's success is often interpreted as one's own failure. As markets mediate such conflicts, these intuitions have been thoroughly investigated in an array of matching models (Roth, 1984, 1985b). Their extent and intensity in dynamic matching markets remains an open question.

To describe agents' collective interests, we let  $\succsim_M$  be the men's *common preference*. Given the matchings  $\mu$  and  $\mu'$ ,  $\mu \succ_M \mu'$  if and only if  $\mu(m) \succ_m \mu'(m)$  for all  $m \in M$  and  $\mu(m) \succ_m \mu'(m)$  for some  $m \in M$ . If  $\mu \succ_M \mu'$  or  $\mu = \mu'$ , then  $\mu \succsim_M \mu'$ . Women's common preference,  $\succsim_W$ , is defined analogously.

When  $T = 1$ , a surprisingly rich set of conclusions has been identified.<sup>20</sup>

(C1) There exists a conflict of interest among agents on opposing sides of the market. If men collectively prefer one stable matching over another, the market's women hold the

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<sup>18</sup>As usual, we call a matching  $\mu$  (*strongly*) *Pareto optimal* if there does not exist a matching  $\mu'$  such that  $\mu'(i) \succsim_i \mu(i)$  for all  $i$  and  $\mu'(i) \succ_i \mu(i)$  for some  $i$ .

<sup>19</sup>In one-period many-to-many matching markets, stable matchings may not be Pareto optimal (Roth and Sotomayor, 1990, Proposition 5.23).

<sup>20</sup>Roth and Sotomayor (1990) provide an overview of these properties.

opposite opinion, i.e. if  $\mu$  and  $\mu'$  are stable, then  $\mu \succsim_M \mu' \iff \mu' \succsim_W \mu$  (Knuth, 1976; Roth, 1985b).

- (C2) Agents on the same side of the market express consensus concerning which stable outcomes are preferable. If  $\mu$  is a stable matching, it is *M-optimal* if for every other stable matching  $\mu'$ ,  $\mu \succsim_M \mu'$ . When  $T = 1$ , the matching identified by the man-proposing deferred acceptance algorithm is *M-optimal* (Gale and Shapley, 1962).
- (C3) Conclusions 1 and 2 stem from the stable set's lattice structure when ordered by  $\succsim_M$  (attributed to John Conway by Knuth, 1976). A *lattice* is a partially-ordered set where any two elements have a greatest lower bound and a least upper bound (Birkhoff, 1940). Thus, welfare comparisons among stable matchings are relatively straightforward.<sup>21</sup>

Once  $T \geq 2$ , the above conclusions do not apply without further qualifications. Theorem 2 showed that a dynamically stable matching may be Pareto-dominated by another dynamically stable matching. Therefore, interests are no longer necessarily opposed. Moreover, as illustrated by the following example, a multi-period market may lack an *M-optimal* matching. Thus, the set of dynamically stable matchings (when ordered by  $\succsim_M$ ) cannot be a lattice.

**Example 3.** There are three men and women. Their preferences are:

$$\succ_{m_1} : w_2w_2, w_2w_1, w_2w_3, w_1w_1, w_3w_3, m_1m_1, \dots$$

$$\succ_{m_2} : w_3w_3, w_3w_2, w_3w_1, w_2w_2, w_1w_1, m_2m_2, \dots$$

$$\succ_{m_3} : w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3m_3, \dots$$

$$\succ_{w_1} : m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, m_3m_3, \dots$$

$$\succ_{w_2} : m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, w_2w_2, m_1m_1, \dots$$

$$\succ_{w_3} : m_1m_1, m_1m_3, m_3m_1, w_3m_1, m_2m_1, m_3m_3, w_3w_3, m_2m_2, \dots$$

In this market there are three dynamically stable matchings (Table 3).  $m_1$  and  $m_2$  like  $\mu^3$  the most.  $m_3$  prefers  $\mu^1$ . Therefore, there does not exist an *M-optimal* stable matching.

To transform the preceding negative conclusions into positive claims we will first focus on a subset of dynamically stable matchings which enjoys a tractable structure. Thereafter,

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<sup>21</sup>Given the stable matchings  $\mu$  and  $\mu'$ , there exists a stable matching  $\mu''$  such that  $\mu'' \succsim_M \mu, \mu'$ . Hence, men gain when the market moves to  $\mu''$  from  $\mu$  or  $\mu'$ . By (C1), women lose from this transition.



Table 3: All dynamically stable matchings in Example 3.

Matching	$m_1$	$m_2$	$m_3$	$w_1$	$w_2$	$w_3$
$\mu^1$	$w_1w_1$	$w_2w_2$	$w_3w_3$	$m_1m_1$	$m_2m_2$	$m_3m_3$
$\mu^2$	$w_3w_3$	$w_1w_1$	$w_2w_2$	$m_2m_2$	$m_3m_3$	$m_1m_1$
$\mu^3$	$w_2w_3$	$w_3w_1$	$w_1w_2$	$m_3m_2$	$m_1m_3$	$m_2m_1$

we will add additional regularity to agents preferences extending that structure to the entire stable set.

### 3.1 Persistent Matchings and Lattice Structures

Fix an economy and let  $\mathbb{D}$  be the set of dynamically stable matchings. We sometimes write  $(\mathbb{D}, \succsim_M)$  to emphasize this set's ordering by  $\succsim_M$ . The matching  $\mu$  is maximally persistent if  $\mu(i)$  is maximally persistent for each  $i$ . Let  $\mathbb{P} \subset \mathbb{D}$  be the set of maximally-persistent, dynamically stable matchings. There is a close connection between  $\mathbb{P}$  and stable matchings in a one-period economy.

**Lemma 3.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i$  for all  $i$ . Let  $\mu$  be a maximally-persistent matching. The matching  $\mu$  is dynamically stable if and only if  $\mu_t$  is a stable matching in a one-period market where agent  $i$ 's preference coincides with  $P_{\succsim_i}$ .*

The applicability of conclusions C1–C3 on this restricted domain follows as a corollary.

**Corollary 1.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i$  for all  $i$ . (i)  $(\mathbb{P}, \succsim_M)$  is a lattice. (ii) The matching identified by the man-proposing E-DA procedure is  $M$ -optimal among matchings in  $\mathbb{P}$ . (iii) For all  $\mu, \mu' \in \mathbb{P}$ ,  $\mu \succsim_M \mu' \iff \mu' \succsim_W \mu$ .*

*Proof.* By Lemma 3, each maximally-persistent, dynamically stable matching in a  $T$ -period economy corresponds to a stable matching in a one-period economy and vice versa. The set of stable matchings in a one-period economy is a lattice when ordered by men's common preference (Knuth, 1976). Thus,  $(\mathbb{P}, \succsim_M)$  is a lattice as well. Points (ii) and (iii) follow from Gale and Shapley (1962) and Knuth (1976), as noted in C1–C3 above.  $\square$

### 3.2 Volatile Matchings and Lattice Structures

To extend the preceding conclusions beyond  $\mathbb{P}$  it is necessary to restrict agents' preferences. In light of Example 3, we first resolve the non-existence of an  $M$ -optimal matching. The

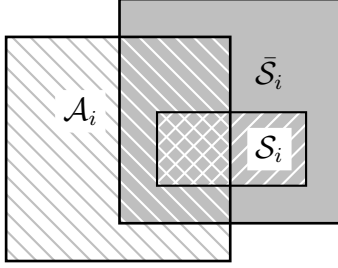


Figure 1: Preference domains.  $\mathcal{S}_i$  – preferences that reflect a spot ranking;  $\bar{\mathcal{S}}_i$  – preferences that exhibit inertia relative to  $\mathcal{S}_i$ ;  $\mathcal{A}_i$  – sacrifice averse preferences.

restriction we propose is the following:

**Definition 8.** The preference  $\succ_i$  is *sacrifice averse* if for all partnership plans  $x$  and  $\bar{j}$ ,  $x \succ_i \bar{j} \implies \bar{x}_t \succsim_i \bar{j}$ . Let  $\mathcal{A}_i$  be the set of preferences for  $i$  that are sacrifice averse.

Figure 1 sketches the relationships among  $\mathcal{S}_i$ ,  $\bar{\mathcal{S}}_i$ , and  $\mathcal{A}_i$ . The following lemma rationalizes the “sacrifice averse” nomenclature though the condition’s implications are broader.<sup>22</sup> In a stable matching, an agent never accepts a partner worse than being single. When  $\succ_i \notin \mathcal{A}_i$ , this may not be true (Example 2).

**Lemma 4.** *If  $\mu \in \mathbb{D}$  and  $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ , then  $\mu_t(i) R_{\succ_i} i$  for all  $t$ .*

When we restrict agents’ preferences to  $\bar{\mathcal{S}}_i \cap \mathcal{A}_i$ , the stable set gains added structure and an  $M$ -optimal stable matching exists.

**Theorem 3.** *Suppose  $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ . The matching identified by the man-proposing  $E$ -DA procedure is  $M$ -optimal.*

We prove Theorem 3 as a corollary to Theorem 4 below.

Given Theorem 3, a tempting conjecture is that when  $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ ,  $(\mathbb{D}, \succ_M)$  is a lattice. Surprisingly, this need not be the case.

<sup>22</sup>In words the condition can read as follows: If a worker prefers successive, short-term contracts at Firm A and then at Firm B in lieu of a stable long-term job at Firm C, then he must prefer long-term employment at Firm A and Firm B over long-term employment at Firm C. Examples 4 and 5 show that this condition *does not* preclude volatile dynamically stable matchings.

**Example 4.** Suppose there are two men and two women. Their preferences are:

$$\succ_{m_1} : w_1 w_1 w_1, w_2 w_1 w_1, w_1 w_1 w_2, w_1 w_2 w_2, w_2 w_2 w_1, w_2 w_2 w_2, \dots$$

$$\succ_{m_2} : w_2 w_2 w_2, w_2 w_2 w_1, w_1 w_2 w_2, w_1 w_1 w_2, w_2 w_1 w_1, w_1 w_1 w_1, \dots$$

$$\succ_{w_1} : m_2 m_2 m_2, m_1 m_2 m_2, m_2 m_2 m_1, m_2 m_1 m_1, m_1 m_1 m_2, m_1 m_1 m_1, \dots$$

$$\succ_{w_2} : m_1 m_1 m_1, m_1 m_1 m_2, m_2 m_1 m_1, m_2 m_2 m_1, m_1 m_2 m_2, m_2 m_2 m_2, \dots$$

This market has six dynamically stable matchings (Table 4).  $(\mathbb{D}, \succ_M)$  is not a lattice since  $\mu^4$  and  $\mu^5$  do not have a unique least upper bound (Figure 2).

Table 4: All dynamically stable matchings in Example 4.

Matching	$m_1$	$m_2$	$w_1$	$w_2$
$\mu^1$	$w_1 w_1 w_1$	$w_2 w_2 w_2$	$m_1 m_1 m_1$	$m_2 m_2 m_2$
$\mu^2$	$w_2 w_1 w_1$	$w_1 w_2 w_2$	$m_2 m_1 m_1$	$m_1 m_2 m_2$
$\mu^3$	$w_1 w_1 w_2$	$w_2 w_2 w_1$	$m_1 m_1 m_2$	$m_2 m_2 m_1$
$\mu^4$	$w_1 w_2 w_2$	$w_2 w_1 w_1$	$m_1 m_2 m_2$	$m_2 m_1 m_1$
$\mu^5$	$w_2 w_2 w_1$	$w_1 w_1 w_2$	$m_2 m_2 m_1$	$m_1 m_1 m_2$
$\mu^6$	$w_2 w_2 w_2$	$w_1 w_1 w_1$	$m_2 m_2 m_2$	$m_1 m_1 m_1$

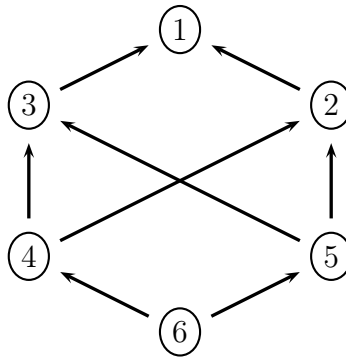


Figure 2: Hasse diagram of  $(\mathbb{D}, \succ_M)$  in Example 4.

Though further restrictions precluding cases like Example 4 can be proposed, we will instead pursue a different vein. Like Blair (1988), who studies a many-to-many matching

market, we will arrange  $\mathbb{D}$  with a weaker partial order. Importantly, the new ordering is derived from on agents' preferences and therefore has a behavioral foundation. Thus, it continues to facilitate meaningful welfare analysis. We define it by first weakening the ordering of partnership plans implied by each agent's preference.

**Definition 9.** Consider agent  $i$  with preference  $\succsim_i$ . Agent  $i$  *decisively prefers plan  $x$  to plan  $y$* , denoted  $x \succsim_i^* y$ , if (i)  $x \succsim_i y$  and (ii)  $\overline{x}_t \succsim_i \overline{y}_{t'}$  for all  $t$  and  $t'$ . We write  $x \sim_i^* y$  when  $x = y$  or  $x \succsim_i^* y$ .

Though we use  $\sim_i^*$  below to define an alternative common preference for each side of the market, at least two additional motivations support its introduction. Both are justifications for the usefulness of incomplete preferences, like  $\sim_i^*$ , in decision analysis.

First, and stepping back from the assumption that  $\succsim_i$  describes  $i$ 's true preferences, we can interpret  $\sim_i^*$  to be  $i$ 's best ranking of available plans after some reflection, but he retains some indecisiveness among similar options (Aumann, 1962). Notably,  $\sim_i^*$  provides a complete ranking of maximally-persistent plans, which seems plausible. However, the agent's arbitration among volatile plans is more clouded. To illustrate, take  $m \in M$  and suppose that

$$w_1w_1 \succsim_m w_1w_2 \succsim_m w_2w_1 \succsim_m w_2w_2 \succsim_m w_1w_3 \succsim_m w_3w_3 \succsim_m w_2w_3. \quad (2)$$

Figure 3 illustrates  $\sim_m^*$  as derived from (2). In this case,  $m$  is certain that  $w_1w_1$  dominates  $w_2w_2$ , which dominates  $w_3w_3$ . However,  $w_1w_2$ ,  $w_2w_1$ , and  $w_1w_3$  are more difficult to compare. Interestingly,  $\sim_m^*$  does not rank  $w_2w_3$  “in between”  $w_2w_2$  and  $w_3w_3$ . Thus,  $\sim_m^*$  accounts for preference inertia— $w_2w_3$  incorporates an assignment to a better partner but  $w_3w_3$  is more persistent. In this case,  $m$  is unsure which is better. When an agent is only certain of  $\sim_i^*$ , we may interpret  $\succsim_i$  to be a complete ranking of available outcomes reported by  $i$  when pressed for such a response.<sup>23</sup>

While the first case places  $\sim_i^*$ 's origin with the agent, an alternative interpretation casts  $\sim_i^*$  as an outside observer's best guess concerning  $i$ 's preferences (Ok, 2002). The observer is unaware of  $i$ 's refined opinions given by  $\succsim_i$  but may be able to use  $\sim_i^*$  as a conservative benchmark for market and welfare analysis.

Whether  $\sim_i^*$  is motivated by behavioral concerns or analytic limitations, it extends to an ordering of matchings in the usual way:  $\mu \succ_M^* \mu'$  if and only if  $\mu(m) \succsim_m^* \mu'(m)$  for all  $m \in M$  and  $\mu(m) \succ_m^* \mu'(m)$  for some  $m \in M$ . The associated weak relation,  $\sim_M^*$ , and the corresponding common preference for women,  $\sim_W^*$ , are defined as expected.

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<sup>23</sup>In this case, stable outcomes would be stable with respect to reported preferences.

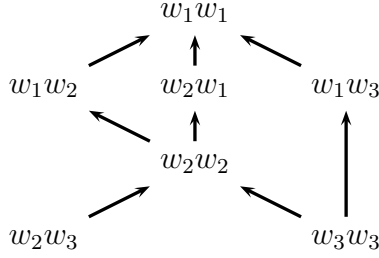


Figure 3: The  $\succsim_m^*$  order derived from (2).

Below, Theorem 4 shows that  $(\mathbb{D}, \succsim_M^*)$  is a lattice. To shorten the exposition, we precede the argument with six preliminary lemmas. Lemmas 5–7 focus on agents’ true preferences,  $\succsim_i$ . Of these claims, Lemma 5 is of independent interest. Coupled with Lemma 3, it says that when  $\succsim_i \in \bar{\mathcal{S}}_i$ , the final period assignment in a dynamically stable matching is a stable assignment in a corresponding one-period economy. This conclusion is similar to the requirement that agents’ final-period actions in a Nash equilibrium of a finitely repeated game are also equilibrium actions in the constituent stage game. Lemmas 8–10 focus on decisive preferences and the  $\succsim_M^* / \succsim_W^*$  orderings. Lemma 8 shows that  $\succsim_M$  and  $\succsim_M^*$  coincide on  $\mathbb{P}$ . Lemma 9 shows that every dynamically stable matching is dominated by a dynamically stable matching in  $\mathbb{P}$ . Finally, Lemma 10 shows that  $\succsim_W^*$  is the inverse of  $\succsim_M^*$  when restricted to  $\mathbb{D}$ . Thus,  $\succsim_M^*$  and  $\succsim_W^*$  characterize the conflict of interest between the market’s two sides, when examining dynamically stable matchings.

**Lemma 5.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i$  for all  $i$ . If  $\mu \in \mathbb{D}$ , then  $\overline{\mu_T} = (\mu_T, \dots, \mu_T) \in \mathbb{D}$ .*

**Lemma 6.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ . If  $\mu \in \mathbb{D}$ , then  $\overline{\mu_t} = (\mu_t, \dots, \mu_t) \in \mathbb{D}$ .*

**Lemma 7.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ . Let  $\mu \in \mathbb{D}$ . (i) If  $\mu(i)$  is volatile, there exists a period  $t(i)$  such that  $\mu(i) \succsim_i \overline{\mu_{t(i)}(i)}$ . (ii) If  $\mu(i) \succsim_i \overline{\mu_{t(i)}(i)}$ , then for all  $t'$  such that  $\mu_{t'}(i) \neq \mu_{t(i)}$ ,  $\overline{\mu_{t'}(i)} \succsim_i \mu(i)$ .*

**Lemma 8.** *Let  $\mu$  and  $\mu'$  be maximally-persistent matchings. For all  $i$ ,  $\mu(i) \succsim_i \mu'(i) \iff \mu(i) \succsim_i^* \mu'(i)$  and  $\mu \succsim_M \mu' \iff \mu \succsim_M^* \mu'$ . Therefore,  $(\mathbb{P}, \succsim_M) = (\mathbb{P}, \succsim_M^*)$ .*

**Lemma 9.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ . If  $\mu \in \mathbb{D}$ ,  $\exists \mu' \in \mathbb{P} \subset \mathbb{D}$  such that  $\mu' \succsim_M^* \mu$ .*

**Lemma 10.** *Suppose  $\succsim_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ . Let  $\mu, \mu' \in \mathbb{D}$ .  $\mu \succsim_M^* \mu' \iff \mu' \succsim_W^* \mu$ .*

The lattice structure of  $(\mathbb{D}, \succ_M^*)$  follows from the preceding claims.

**Theorem 4.** *Suppose  $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$  for all  $i$ .  $(\mathbb{D}, \succ_M^*)$  is a lattice.*

*Proof.* By Lemma 10,  $\succ_W^*$  is the inverse of  $\succ_M^*$  on  $\mathbb{D}$ . Thus, to verify that  $(\mathbb{D}, \succ_M^*)$  is a lattice it is sufficient to show that there exists a least upper bound for any two dynamically stable matchings.

Let  $\mu, \mu' \in \mathbb{D}$ . If  $\mu \succ_M^* \mu'$ , then  $\mu$  is the least upper bound for  $\mu$  and  $\mu'$ . Instead, suppose  $\mu$  and  $\mu'$  are not ordered by  $\succ_M^*$ . From Lemma 9 we know that  $\mu$  and  $\mu'$  are bounded above by some  $\mu^1, \mu^2 \in \mathbb{D}$ , i.e.

$$\left. \begin{array}{l} \mu^1 \\ \mu^2 \end{array} \right\} \succ_M^* \left\{ \begin{array}{l} \mu \\ \mu' \end{array} \right\}.$$

It is sufficient to show that  $\exists \tilde{\mu} \in \mathbb{D}$  such that

$$\left. \begin{array}{l} \mu^1 \\ \mu^2 \end{array} \right\} \succ_M^* \tilde{\mu} \succ_M^* \left\{ \begin{array}{l} \mu \\ \mu' \end{array} \right\}.$$

Let  $\mathbb{P}(\mu^k) = \{\overline{\mu_t^k} : t = 1, \dots, T\}$ . By Lemma 6,  $\overline{\mu_t^k} \in \mathbb{P}$ . Since  $(\mathbb{P}, \succ_M^*)$  is a lattice,  $\mathbb{P}(\mu^1) \cup \mathbb{P}(\mu^2) \subset \mathbb{P}$  has a greatest lower bound in  $(\mathbb{P}, \succ_M^*)$ . Let  $\lambda \in \mathbb{P}$  be this greatest lower bound.

Next, we argue that for each  $m$ ,  $\mu^k \succ_m^* \lambda$ . There are two properties to check:

1. For each  $t$  and  $t'$ ,  $\overline{\mu_t^k(m)} \succ_m \overline{\lambda_{t'}(m)}$ .

$\overline{\mu_t^k} \in \mathbb{P}(\mu^k)$ . Hence,  $\overline{\mu_t^k} \succ_M^* \lambda$ . This implies  $\overline{\mu_t^k(m)} \succ_m^* \lambda(m) = \overline{\lambda_{t'}(m)}$  for all  $t'$  as  $\lambda \in \mathbb{P}$ .

2.  $\mu^k(m) \succ_m \lambda(m)$ .

Assume the contrary. Then  $\lambda(m) \succ_m \mu^k(m)$ . From part (1) above we know that for all  $t$ ,  $\overline{\mu_t^k(m)} \succ_m \tilde{\mu}(m) \succ_m \mu^k(m)$ . There are two possibilities. If  $\mu^k(m) \in \mathbb{P}$ , then  $\mu^k(m) = \overline{\mu_t^k(m)}$  and we have a contradiction. If instead  $\mu^k \notin \mathbb{P}$ , then by Lemma 7  $\mu^k(m) \succ_m \overline{\mu_t^k(m)}$  for some  $t$ . This too is a contradiction. Hence, our assumption to the contrary was incorrect.

To complete the proof, we verify that  $\lambda(m) \succ_m^* \mu(m)$  for all  $m$ . (The case of  $\mu'$  follows identically.) As a preliminary point we note that since  $\mu^k \succ_M^* \mu$ , for all  $m, t, t'$ , and  $k$ ,  $\overline{\mu_t^k(m)} \succ_m \overline{\mu_{t'}(m)}$ . Suppose that for some  $m$ ,  $\lambda(m) \not\succeq_m^* \mu(m)$ . There are two possibilities:

1. For some  $t$ ,  $\overline{\mu_t(m)} \succ_m \lambda(m)$ .

Given the observation made immediately above,  $\overline{\mu_t} \in \mathbb{P}$  is a lower bound for  $\mathbb{P}(\mu^1) \cup \mathbb{P}(\mu^2)$ . But  $\lambda$  is the greatest such lower bound. Therefore, this case cannot be true.

2.  $\mu(m) \succ_m \lambda(m)$ .

If  $\mu(m) \in \mathbb{P}$ , then case (1) above applies. If  $\mu(m) \notin \mathbb{P}$ , then by Lemma 7  $\overline{\mu_t(m)} \succ_m \lambda(m)$  for some  $t$ . But again, this implies case (1) applies.

As neither case (1) nor case (2) applies, we conclude that in fact  $\lambda(m) \succ_m^* \mu(m) \forall m$ .  $\square$

A corollary to Theorem 4 is that an  $M$ -optimal matching exists (Theorem 3). If  $\mu \in \mathbb{D}$ , then there exists  $\mu^* \in \mathbb{P}$  such that  $\mu^* \succ_M^* \mu \implies \mu^*(m) \succ_m^* \mu(m) \forall m \implies \mu^*(m) \succ_m \mu(m) \forall m \implies \mu^*(m) \succ_M \mu(m)$ . Thus, the  $\succ_M$ -maximal matching in  $\mathbb{P}$ , which is identified by the E-DA procedure, dominates all matchings in  $\mathbb{D}$  and is  $M$ -optimal.

The following example illustrates how  $\succ_M^*$  and  $\succ_W^*$  rearrange the stable set, as claimed. The conflict of interest between the market's two sides also becomes apparent.

**Example 5.** Let  $T = 2$ . There are four men and women with the following preferences:

$$\succ_{m_1} : w_1w_1, w_4w_4, w_1w_4, w_4w_1, w_2w_1, w_2w_4, w_1w_2, w_2w_2, w_3w_3, \dots$$

$$\succ_{m_2} : w_2w_2, w_3w_3, w_2w_3, w_3w_2, w_1w_2, w_1w_3, w_2w_1, w_1w_1, w_4w_4, \dots$$

$$\succ_{m_3} : w_3w_3, w_2w_2, w_3w_2, w_2w_3, w_4w_3, w_4w_2, w_3w_4, w_4w_4, w_1w_1, \dots$$

$$\succ_{m_4} : w_4w_4, w_1w_1, w_4w_1, w_1w_4, w_3w_4, w_3w_1, w_4w_3, w_3w_3, w_2w_2, \dots$$

$$\succ_{w_1} : m_2m_2, m_3m_3, m_2m_3, m_3m_2, m_2m_1, m_3m_1, m_1m_2, m_1m_1, m_4m_4, \dots$$

$$\succ_{w_2} : m_1m_1, m_4m_4, m_1m_4, m_4m_1, m_1m_2, m_4m_2, m_2m_1, m_2m_2, m_3m_3, \dots$$

$$\succ_{w_3} : m_4m_4, m_1m_1, m_4m_1, m_1m_4, m_4m_3, m_1m_3, m_3m_4, m_3m_3, m_2m_2, \dots$$

$$\succ_{w_4} : m_3m_3, m_2m_2, m_3m_2, m_2m_3, m_3m_4, m_2m_4, m_4m_3, m_4m_4, m_1m_1, \dots$$

This market has 16 dynamically stable matchings (Table 5). Figure 4 summarizes these matchings when ordered by  $\succ_M$  and by  $\succ_W$ .  $\succ_W$  is not the inverse of  $\succ_M$  since a conflict of interest is not maintained throughout the set of dynamically stable outcomes. For instance,  $\mu^6$  strictly Pareto dominates  $\mu^{11}$ . (This proves Theorem 2.) When ordered by  $\succ_M^*$  or  $\succ_W^*$ , the set of dynamically stable matchings is a lattice and  $\succ_W^*$  is the inverse of  $\succ_M^*$  (Figure 5).

Table 5: All dynamically stable matchings in Example 5.

Matching	$m_1$	$m_2$	$m_3$	$m_4$	$w_1$	$w_2$	$w_3$	$w_4$
$\mu^1$	$w_1w_1$	$w_2w_2$	$w_3w_3$	$w_4w_4$	$m_1m_1$	$m_2m_2$	$m_3m_3$	$m_4m_4$
$\mu^2$	$w_1w_1$	$w_2w_2$	$w_4w_3$	$w_3w_4$	$m_1m_1$	$m_2m_2$	$m_4m_3$	$m_3m_4$
$\mu^3$	$w_1w_1$	$w_2w_2$	$w_3w_4$	$w_4w_3$	$m_1m_1$	$m_2m_2$	$m_3m_4$	$m_4m_3$
$\mu^4$	$w_1w_1$	$w_2w_2$	$w_4w_4$	$w_3w_3$	$m_1m_1$	$m_2m_2$	$m_4m_4$	$m_3m_3$
$\mu^5$	$w_2w_1$	$w_1w_2$	$w_3w_3$	$w_4w_4$	$m_2m_1$	$m_1m_2$	$m_3m_3$	$m_4m_4$
$\mu^6$	$w_2w_1$	$w_1w_2$	$w_4w_3$	$w_3w_4$	$m_2m_1$	$m_1m_2$	$m_4m_3$	$m_3m_4$
$\mu^7$	$w_2w_1$	$w_1w_2$	$w_3w_4$	$w_4w_3$	$m_2m_1$	$m_1m_2$	$m_3m_4$	$m_4m_3$
$\mu^8$	$w_2w_1$	$w_1w_2$	$w_4w_4$	$w_3w_3$	$m_2m_1$	$m_1m_2$	$m_4m_4$	$m_3m_3$
$\mu^9$	$w_1w_2$	$w_2w_1$	$w_3w_3$	$w_4w_4$	$m_1m_2$	$m_2m_1$	$m_3m_3$	$m_4m_4$
$\mu^{10}$	$w_1w_2$	$w_2w_1$	$w_4w_3$	$w_3w_4$	$m_1m_2$	$m_2m_1$	$m_4m_3$	$m_3m_4$
$\mu^{11}$	$w_1w_2$	$w_2w_1$	$w_3w_4$	$w_4w_3$	$m_1m_2$	$m_2m_1$	$m_3m_4$	$m_4m_3$
$\mu^{12}$	$w_1w_2$	$w_2w_1$	$w_4w_4$	$w_3w_3$	$m_1m_2$	$m_2m_1$	$m_4m_4$	$m_3m_3$
$\mu^{13}$	$w_2w_2$	$w_1w_1$	$w_3w_3$	$w_4w_4$	$m_2m_2$	$m_1m_1$	$m_3m_3$	$m_4m_4$
$\mu^{14}$	$w_2w_2$	$w_1w_1$	$w_4w_3$	$w_3w_4$	$m_2m_2$	$m_1m_1$	$m_4m_3$	$m_3m_4$
$\mu^{15}$	$w_2w_2$	$w_1w_1$	$w_3w_4$	$w_4w_3$	$m_2m_2$	$m_1m_1$	$m_3m_4$	$m_4m_3$
$\mu^{16}$	$w_2w_2$	$w_1w_1$	$w_4w_4$	$w_3w_3$	$m_2m_2$	$m_1m_1$	$m_4m_4$	$m_3m_3$

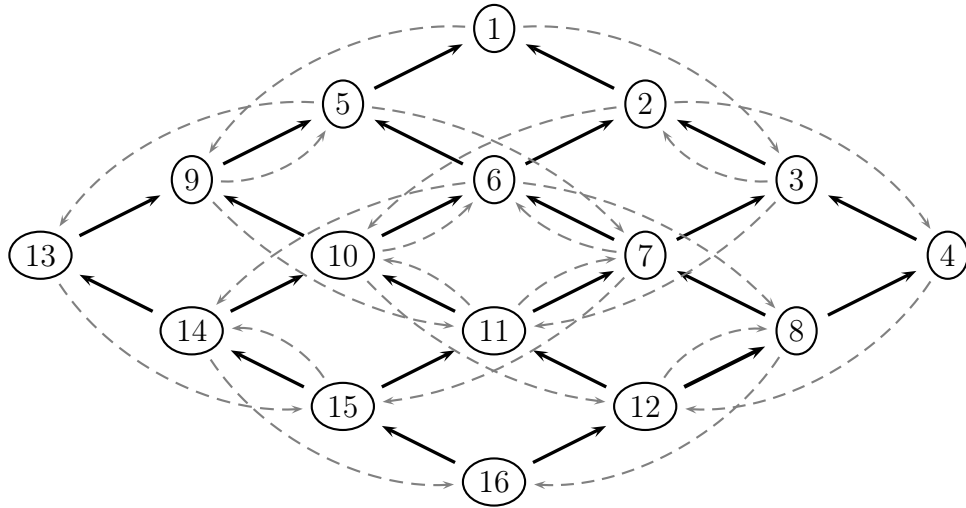


Figure 4: The set of dynamically stable matchings in Example 5 ordered by  $\succ_M$  (solid) and  $\succ_W$  (dashed).



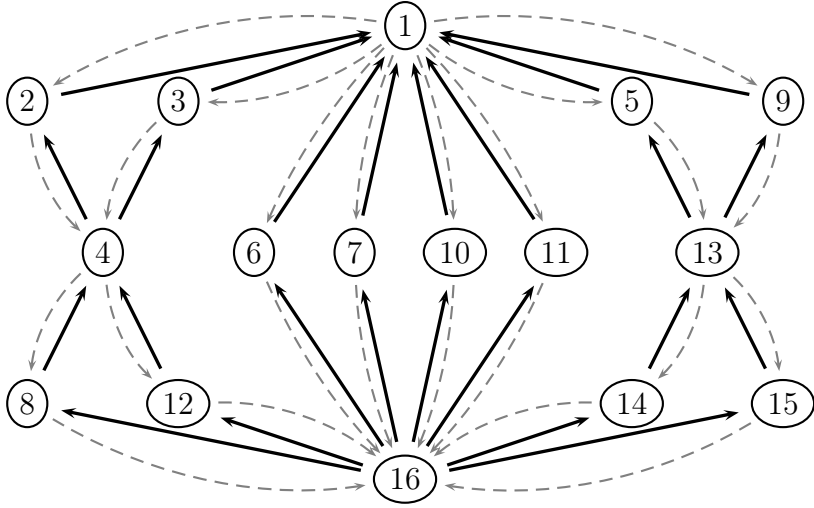


Figure 5: The set of dynamically stable matchings in Example 5 ordered by  $\succ_M^*$  (solid) and  $\succ_W^*$  (dashed).

## 4 Temporal Robustness

We have thus far taken the market’s time horizon, defined by the parameter  $T$ , as given. A natural question, however, concerns the sensitivity of the set of dynamically stable matchings to changes in  $T$ . While some applications of matching models have an “obvious” time horizon and a well-specified set of rematching opportunities, many do not. Only a best assessment guides analysis. Thus, it is important to know whether a particular dynamically stable matching has an easily-identifiable counterpart as the time horizon is slightly perturbed or the frequency of revision opportunities is adjusted.<sup>24</sup>

### 4.1 Abridgment

Comparing markets with different time horizons involves mapping stable matchings from one case to the other. Thus, the set of agents must be the same and agents’ preferences need to be coherently related as the time horizon changes. In this regard, abridgments are simple because matchings can be truncated and preferences can be projected onto a restricted domain. Consider, for example, a  $T$ -period market where  $x = x_1 \cdots x_T$  is a typical

<sup>24</sup>By rescaling time, both cases are the same type of problem.

partnership plan. Agent  $i$ 's preference conditional on  $x_{\leq t}$ , denoted  $\succ_i^{x_{\leq t}}$ , is a preference over plans of length  $T - t$  such that

$$(y_1, \dots, y_{T-t}) \succ_i^{x_{\leq t}} (z_1, \dots, z_{T-t}) \iff (x_{\leq t}, y_1, \dots, y_{T-t}) \succ_i (x_{\leq t}, z_1, \dots, z_{T-t}).$$

By conditioning preferences on the market's elapsed history, we can relate the final periods of a stable outcome given a longer time horizon to a stable matching in a shorter market.

**Theorem 5.** *Let  $\mu^*$  be a dynamically stable matching in a  $T$ -period economy where each agent's preference is  $\succ_i$ . Then  $\mu_{>t}^*$  is a dynamically stable matching in a  $T-t$ -period economy where each agent's preference is  $\succ_i^{\mu_{\leq t}^*(i)}$ .*

A natural follow-up question asks whether stable matchings can be truncated conditioning on the future, rather than the past? As above, agent  $i$ 's preference anticipating  $x_{>t}$ , denoted  $\succ_i^{x_{>t}}$ , is a preference over plans of length  $t$  such that

$$(y_1, \dots, y_t) \succ_i^{x_{>t}} (z_1, \dots, z_t) \iff (y_1, \dots, y_t, x_{>t}) \succ_i (z_1, \dots, z_t, x_{>t}).$$

In this case, Theorem 5's analogue does not hold. The  $t$ -period matching  $\mu_{\leq t}^*$  need not be dynamically stable when preferences are  $\succ_i^{\mu_{>t}^*(i)}$ . As a counterexample,  $m_1$  can block  $\mu_{\leq 2}^3$  in Example 2.

## 4.2 Extension

While every  $T$ -period economy can be shortened, introducing additional periods requires greater finesse. An extension involves embedding the set of dynamically stable matchings into an economy with a longer time horizon. Thus, extensions serve as a robustness check confirming that insights from small- $T$  examples continue to apply when  $T$  is large.

Consider a  $T$ -period economy that now runs for one additional period. To study this new situation, we first extend agents' preferences to account for the extra period.

**Definition 10.** The preference  $\hat{\succ}_i$  is a *one-period extension* of  $\succ_i$  if

$$x_1 \cdots x_T \succ_i y_1 \cdots y_T \iff x_1 \cdots x_T x_T \hat{\succ}_i y_1 \cdots y_T y_T.$$

As a complementary motivation for Definition 10, consider an economy that last two periods of calendar time. However, institutional or legal constraints do not accommodate re-matching between periods. Of course, this restricted economy is equivalent to a one period

economy as the two periods are treated as one. Definition 10 is analogous to introducing a division between the two phases. To illustrate with a canonical application, a student may express the preference that School A is better than School B *if* he must be enrolled in one school for four years. Suppose, however, that the student is allowed to change schools after two years. Given his opinion in the restricted market, it is reasonable to expect the student to rank a plan where he spends the first two years and the last two years at School A ahead of a plan where he spends all years at School B.

Definition 10 imposes relatively minor restrictions on admissible preference extensions. An arbitrary extension may therefore negate a preference's important analytic properties, such as membership in  $\bar{\mathcal{S}}_i$ , for example. Some extensions, however, do preserve such properties and they play a key role in the proof of Theorem 6 below.

**Lemma 11.** *Let  $\mathcal{S}_i$  and  $\hat{\mathcal{S}}_i$  be the sets of preferences that reflect a spot ranking in a  $T$ -period and a  $\hat{T} = T + 1$  -period market, respectively.*

1. *If  $\succ_i \in \mathcal{S}_i$ , then there exists a one-period extension of  $\succ_i$  such that  $\hat{\succ}_i \in \hat{\mathcal{S}}_i$ .*
2. *If  $\succ_i \in \bar{\mathcal{S}}_i$ , then there exists a one-period extension of  $\succ_i$  such that  $\hat{\succ}_i \in \bar{\hat{\mathcal{S}}}_i$ .*

*Proof.* See Appendix A. A lexicographic construction proves (1). A more elaborate construction is required to verify (2) since the dual goals of maintaining a link to a spot ranking while accommodating inertia need to be addressed.  $\square$

**Theorem 6.** *Let  $\mu^* = (\mu_1^*, \dots, \mu_T^*)$  be a dynamically stable matching in a  $T$ -period market where  $\succ_i \in \bar{\mathcal{S}}_i$  for all  $i$ . Consider a  $\hat{T} = T + 1$  -period market with the same set of agents and where each agent's preference  $\hat{\succ}_i \in \bar{\hat{\mathcal{S}}}_i$  is a one-period extension of  $\succ_i$ . The matching  $\hat{\mu}^* = (\mu_1^*, \dots, \mu_T^*, \mu_T^*)$  is dynamically stable in the  $\hat{T}$ -period market.*

Theorem 6 embeds all dynamically stable matchings from a  $T$ -period market into the set of stable matchings from a  $T + 1$ -period market. The converse association is generally not possible. Stable outcomes in longer markets may not have an obvious precursor at a shorter time horizon. To illustrate we adapt an example proposed by Kadam and Kotowski (2015).

**Example 6.** Let  $M = \{m_1, m_2\}$  and  $W = \{w_1, w_2\}$ . Consider a one-period market where agents' preferences are

$$\begin{array}{ll} \succ_{m_1} : w_1, m_1, w_2 & \succ_{w_1} : m_2, m_1, w_1 \\ \succ_{m_2} : w_2, m_2, w_1 & \succ_{w_2} : m_1, m_2, w_2 \end{array}$$

This market has one stable matching where  $\mu^*(m_1) = w_1$  and  $\mu^*(m_2) = w_2$ .

Consider a one-period extension where  $\hat{\succ}_i \in \bar{\mathcal{S}}_i$  for all  $i$ . The agents' preferences are

$$\hat{\succ}_{m_1} : w_1w_1, w_1m_1, w_1w_2, m_1w_1, w_2w_1, m_1m_1, w_2w_2, \dots$$

$$\hat{\succ}_{m_2} : w_2w_2, w_2m_2, w_2w_1, m_2w_2, w_1w_2, m_2m_2, w_1w_1, \dots$$

$$\hat{\succ}_{w_1} : m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1, \dots$$

$$\hat{\succ}_{w_2} : m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2, \dots$$

There are two dynamically stable matchings in this two-period market (Table 6). The matching  $\hat{\mu}^1$  extends  $\mu^*$  in the sense of Theorem 6.  $\hat{\mu}^2$  is not the extension of any stable matching from the one-period economy.

Table 6: All dynamically stable matchings in Example 6.

Matching	$m_1$	$m_2$	$w_1$	$w_2$
$\hat{\mu}^1$	$w_1w_1$	$w_2w_2$	$m_1m_1$	$m_2m_2$
$\hat{\mu}^2$	$w_2w_1$	$w_1w_2$	$m_2m_1$	$m_1m_2$

### 4.3 Discussion

Beyond settling questions of robustness, temporal manipulations also have normative applications worthy of emphasis. To illustrate, consider the problem faced by a school district wishing to match students to schools. Recently, many districts have adopted centralized matching procedures to navigate the conflict between schools' physical capacity constraints and students' preferences (Abdulkadiroğlu and Sönmez, 2003). Such problems have a multi-period nature because each student is assigned to a particular school for many years. Though we often emphasize mechanisms that assign students to a school for a focal duration, such as all four years of high school, it is possible to build-in planned revision opportunities at finer time scales. For instance, a student may like a particular school only because it offers a specialized upper-year curriculum and may prefer or be willing to enroll elsewhere in early years. Allowing him to do so in a pre-planned manner may ease a capacity constraint

and may better align with his preferences.<sup>25</sup> Under the assumptions of Theorem 6, the introduction of additional re-matching opportunities need not beget instability. The original assignment has a natural analogue under the new regime. This conclusion runs counter to the mechanical effect that more blocking opportunities heighten instability.

Welfare improvements stemming from the fine-tuning of rematching frequencies are possible too. For instance, in Example 6 the introduction of a second period allows for a novel stable matching,  $\hat{\mu}^2$ , that the market’s women prefer to the original outcome’s naive extension,  $\hat{\mu}^1$ . Though  $\hat{\mu}^2$  involves a welfare loss for the men relative to  $\hat{\mu}^1$ , this tradeoff may be acceptable in applications. In many school-assignment problems the schools’ “preferences” are administratively-defined “priorities,” which do not have the standard welfare interpretation. Though illustrative, the preceding discussion emphasizes that the allowed or the anticipated revision frequency and the duration of proposed assignments are manipulable parameters in many market-design applications. Similar principles apply to job-assignment problems or to other matchings accommodating a rotation among distinct positions.

## 5 Concluding Remarks

Dynamics are an integral feature of many bilateral markets. Though our model is stylized, it accommodates many properties of real-world interactions that unfold over a nontrivial time horizon. While we have maintained Gale and Shapley’s (1962) original terminology by speaking of a matching between men and women, the theory developed above captures key features of many labor markets and it can be adapted to tackle important allocation problems, such as student-school assignment. We hope the preceding analysis provides a stepping stone toward these applications.

Our analysis points to at least two directions for further research. First, there is considerable scope to refine and extend our analysis’ normative implications. The ability to fine-tune relationship length and re-matching frequency may be useful tools in market-design applications. Second, we have focused on “small markets,” in the economic sense of the term. An analysis of large markets with an eye toward dynamics is necessary to better link our conclusions with those obtained in parallel matching literatures, such as those stressing search behavior.<sup>26</sup> In the latter, it is known that equilibria may not be Pareto optimal due to

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<sup>25</sup>Though not a designed market per se, a parallel phenomenon occurs at the tertiary education level. Some students may plan to initially attend a local community college and then transfer to a major university to complete their degree.

<sup>26</sup>In a recent working paper, Kennes et al. (2014b) examine a “large” dynamic matching market focusing

coordination failures or externalities. We believe that the results presented above, including the qualifications surrounding the lattice of stable outcomes, hint at these implications but more research is required.

## References

- Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. *American Economic Review*, 93(3):729–747.
- Alkan, A. (2001). On preferences over subsets and the lattice structure of stable matchings. *Review of Economic Design*, 6(1):99–111.
- Alkan, A. (2002). A class of multipartner matching markets with a strong lattice property. *Economic Theory*, 19(4):737–746.
- Aumann, R. J. (1962). Utility theory without the completeness axiom. *Econometrica*, 30(3):445–462.
- Birkhoff, G. (1940). *Lattice Theory*. American Mathematical Society, New York.
- Blair, C. (1984). Every finite distributive lattice is a set of stable matchings. *Journal of Combinatorial Theory, Series A*, 37(3):353–356.
- Blair, C. (1988). The lattice structure of the set of stable matchings with multiple partners. *Mathematics of Operations Research*, 13(4):619–628.
- Crawford, V. P. and Knoer, E. M. (1981). Job matching with heterogenous firms and workers. *Econometrica*, 49(2):437–450.
- Damiano, E. and Lam, R. (2005). Stability in dynamic matching markets. *Games and Economic Behavior*, 52(1):34–53.
- Doval, L. (2014). A theory of stability in dynamic matching markets. Mimeo.
- Dur, U. (2012). Dynamic school choice problem. Mimeo.
- Fallick, B., Fleischman, C. A., and Rebitzer, J. B. (2006). Job-hopping in silicon valley: Some evidence concerning the microfoundations of a high-technology cluster. *Review of Economics and Statistics*, 88(3):472–481.
- Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15.

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on agents' strategic incentives.

- Hatfield, J. W. and Kominers, S. D. (2012). Contract design and stability in many-to-many matching. Mimeo.
- Hatfield, J. W. and Milgrom, P. R. (2005). Matching with contracts. *American Economic Review*, 95(4):913–935.
- Kadam, S. V. and Kotowski, M. H. (2015). Multi-period matching. Mimeo.
- Kelso, Jr., A. S. and Crawford, V. P. (1982). Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504.
- Kennes, J., Monte, D., and Tumennasan, N. (2014a). The daycare assignment: A dynamic matching problem. *American Economic Journal: Microeconomics*, 6(4):362–406.
- Kennes, J., Monte, D., and Tumennasan, N. (2014b). Dynamic matching markets and the deferred acceptance mechanism. Mimeo.
- Knuth, D. E. (1976). *Marriage Stables*. University of Montreal Press, Montreal.
- Kreider, R. M. and Ellis, R. (2011). Number, timing and duration of marriages and divorces: 2009. Current Population Reports P70-125, U.S. Census Bureau, Washington, DC.
- Kurino, M. (2009). Credibility, efficiency, and stability: A theory of dynamic matching markets. Mimeo.
- Kurino, M. (2014). House allocation with overlapping generations. *American Economic Journal: Microeconomics*, 6(1):258–289.
- Mortensen, D. T. and Pissarides, C. A. (1994). Job creation and job destruction in the theory of unemployment. *Review of Economic Studies*, 61(3):397–415.
- Ok, E. A. (2002). Utility representation of an incomplete preference relation. *Journal of Economic Theory*, 104(2):429–449.
- Pereyra, J. S. (2013). A dynamic school choice model. *Games and Economic Behavior*, 80:100–114.
- Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica*, 52(1):47–58.
- Roth, A. E. (1985a). The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory*, 36(2):277–288.
- Roth, A. E. (1985b). Conflict and coincidence of interest in job matching: Some new results and open questions. *Mathematics of Operations Research*, 10(3):379–389.
- Roth, A. E. (1986). On the allocation of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica*, 54(2):425–427.

- Roth, A. E. (2008). Deferred acceptance algorithms: History, theory, practice, and open questions. *International Journal of Game Theory*, 36(3–4):537–569.
- Roth, A. E. and Sotomayor, M. A. O. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press, New York.
- Samuelson, P. A. (1937). A note on measurement of utility. *Review of Economic Studies*, 4(2):155–161.
- Samuelson, W. F. and Zeckhauser, R. J. (1988). Status quo bias in decision making. *Journal of Risk and Uncertainty*, 1(1):7–59.
- Sotomayor, M. (1999). The lattice structure of the set of stable outcomes of the multiple partners assignment game. *International Journal of Game Theory*, 28(4):567–583.
- Stetser, M. C. and Stillwell, R. (2014). *Public High School Four-Year On-Time Graduation Rates and Event Dropout Rates: School Years 2010–11 and 2011–12*. U.S. Department of Education, Washington, DC.
- U.S. Bureau of Labor Statistics (2012). Number of jobs held, labor market activity, and earnings growth among the youngest baby boomers: Results from a longitudinal survey. News Release USDL-12-1489.



## A Appendix: Proofs

**Proof of Lemma 1.** ( $\Rightarrow$ ) Suppose  $x \succ'_i y$ . Since  $\succ_i \in \Upsilon(\succ'_i)$  and  $x \supseteq y$ , Definition 7 implies that  $x \succ_i y$ . ( $\Leftarrow$ ) Suppose  $x \succ_i y$ . Therefore,  $x \neq y$ . If  $y \succ'_i x$ , then  $y \supseteq x$  and  $\succ_i \in \Upsilon(\succ'_i)$  imply that  $y \succ_i x$ —a contradiction. Therefore,  $x \succ'_i y$ .  $\square$

Lemmas A.1 and A.2 are preliminary results used in some of the arguments to follow.

**Lemma A.1.** *Suppose  $\succ_i \in \bar{\mathcal{S}}_i$  and let  $x$  be a volatile partnership plan.  $\bar{x}_t \succ_i x$  for some  $t$ .*

**Proof of Lemma A.1.** Suppose the contrary. Since  $\succ_i \in \bar{\mathcal{S}}_i$ ,  $\succ_i \in \Upsilon(\succ'_i)$  for some  $\succ'_i \in \mathcal{S}_i$ . Because  $\bar{x}_t \supset x$ ,  $x \succ_i \bar{x}_t \implies x \succ'_i \bar{x}_t$  for all  $t$ . Let  $x_{t^*}$  be such that  $x_{t^*} R_{\succ'_i} x_t \forall t$ .  $x \succ_i \bar{x}_{t^*}$  implies that  $x_t P_{\succ_i} x_{t^*}$  for some  $t \neq t^*$ , which is a contradiction.  $\square$

**Lemma A.2.** *Suppose  $\succ_i \in \bar{\mathcal{S}}_i$ . Suppose agent  $i$  can period- $t$  block the matching  $\mu$  for some  $t \geq 2$ . (i) If  $\mu(i)$  is maximally persistent, then  $i$  can period-1 block  $\mu$ . (ii) If  $\mu(i)$  is volatile, then  $i$  can also period- $t'$  block  $\mu$  where  $\mu_{t'-1}(i) \neq \mu_{t'}(i)$ .*

**Proof of Lemma A.2.** (i) Suppose  $\mu(i) = \bar{j}$  for some  $j \neq i$ . Suppose  $i$  can period- $t$  block  $\mu$  but cannot period-1 block  $\mu$ . Then  $(\bar{j}_{<t}, \bar{i}_{\geq t}) \succ_i \bar{j} \succ_i \bar{i}$ . Since  $\bar{i} \supseteq (\bar{j}_{<t}, \bar{i}_{\geq t})$ , this implies  $(\bar{j}_{<t}, \bar{i}_{\geq t}) \succ'_i \bar{i}$  for some  $\succ'_i \in \mathcal{S}_i$  such that  $\succ_i \in \Upsilon(\succ'_i)$ . But this implies  $j P_{\succ_i} i \implies \bar{j} \succ'_i (\bar{j}_{<t}, \bar{i}_{\geq t})$ . As  $\bar{j} \supseteq (\bar{j}_{<t}, \bar{i}_{\geq t})$ ,  $\bar{j} \succ_i (\bar{j}_{<t}, \bar{i}_{\geq t})$ , which is a contradiction.

(ii) Suppose  $\mu_{t-1}(i) = \mu_t(i)$ . (Otherwise  $t = t'$  satisfies the lemma's claim.) Without loss of generality, we can write  $\mu(i)$  as

$$\mu(i) = (\mu_{<s}(i), \bar{j}_{[s,t]}, \underset{\uparrow}{j}, \bar{j}_{(t,s')}, \mu_{>s'}(i)).$$

where  $\mu_{s-1}(i) \neq j$  and  $\mu_{s'+1}(i) \neq j$ . By assumption  $s < t$ . If we let

$$\tilde{\mu}(i) = (\mu_{<s}(i), \bar{j}_{[s,t]}, \underset{\uparrow}{i}, \bar{i}_{(t,s')}, \bar{i}_{>s'})$$

then by assumption  $\tilde{\mu}(i) \succ_i \mu(i)$ . There are three cases.

1. If  $j = i$ , then  $t' = s$ .
2. If  $j P_{\succ_i} i$ , then  $(\mu_{<s}(i), \bar{j}_{[s,t]}, j, \bar{j}_{(t,s')}, \bar{i}_{>s'}) \succ'_i \tilde{\mu}(i)$ . By inspection,  $\tilde{\mu}(i) \parallel \mu(i)$ . Hence,  $(\mu_{<s}(i), \bar{j}_{[s,t]}, j, \bar{j}_{(t,s')}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i)$ , which implies  $(\mu_{<s}(i), \bar{j}_{[s,t]}, j, \bar{j}_{(t,s')}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i) \succ_i \mu(i)$ . Thus,  $t' = s' + 1$ .

3. If  $iP_{\succ_i}j$ , then  $(\mu_{<s}(i), \bar{i}_{[s,t]}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \succ'_i \tilde{\mu}(i)$ . Since  $(\mu_{<s}(i), \bar{i}_{[s,t]}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \supseteq \tilde{\mu}(i)$ ,  $(\mu_{<s}(i), \bar{i}_{[s,t]}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i) \succ_i \mu(i)$ . Thus,  $t' = s$ .

□

**Proof of Lemma 2.** By assumption,  $\hat{\mu}(i) = (\mu_{<t}(i), \mu_{\geq t}^{\{m,w\}}(i)) \succ_i \mu(i)$  for  $i \in \{m, w\}$ . As  $\mu$  cannot be blocked by  $m$  or  $w$  alone, there exists  $t' \geq t$  such that  $\hat{\mu}_{t'}(m) = w$  and  $\hat{\mu}_{t'}(w) = m$ . Let  $t'$  be the smallest such index. Given  $t'$  define  $\tilde{\mu}(m) \equiv (\mu_{<t}(m), \hat{\mu}_{[t,t']}(m), \bar{w}_{\geq t'})$ . By construction,  $\tilde{\mu}(m) \supseteq \hat{\mu}(m)$ .

Since  $\succ_i \in \Upsilon(\succ'_i)$  for some  $\succ'_i \in \mathcal{S}_i$ ,  $(\mu_{<t}(m), \bar{m}_{\geq t}) \supseteq \hat{\mu}(m)$  and  $\hat{\mu}(m) \succ_m \mu(m) \succ_m (\mu_{<t}(m), \bar{m}_{\geq t})$  imply that  $\hat{\mu}(m) \succ'_m (\mu_{<t}(m), \bar{m}_{\geq t})$ . Hence,  $wP_{\succ_m}m$ . Thus,  $\tilde{\mu}(m) \succ'_m \hat{\mu}(m)$  and, therefore,  $\tilde{\mu}(m) \succ_m \hat{\mu}(m)$ . Likewise, we conclude that  $\tilde{\mu}(w) \succ_w \hat{\mu}(w)$ .

If  $t' = t$ , then the proof is complete. Suppose  $t' > t$ . In this case  $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ'_m \tilde{\mu}(m)$ . There are two cases:

1. If  $(\mu_{<t}(m), \bar{w}_{\geq t}) \supseteq \tilde{\mu}(m)$ , then  $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m)$ .
2. If  $(\mu_{<t}(m), \bar{w}_{\geq t}) \not\supseteq \tilde{\mu}(m)$ , then  $\tilde{\mu}(m) \not\supseteq (\mu_{<t}(m), \bar{w}_{\geq t})$ .<sup>27</sup> Hence,  $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m)$ .

In both cases we see that  $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m) \succ_m \hat{\mu}(m) \succ_m \mu(m)$ . An analogous argument applies to  $w$ . □

**Lemma A.3.** Suppose  $\succ_i \in \bar{\mathcal{S}}_i$  for all  $i$ . Let  $\mu$  be a dynamically stable matching when  $T = 2$ . Then agent  $i$  either has a partner in each period or remains unmatched in both periods.

*Proof.* Let  $\mu$  be a dynamically stable matching. Without loss of generality suppose  $m_1 \in M$  is single in period 1 and matched to  $w_1 \in W$  in period 2. (The argument when  $m_1$  is matched to  $w_1$  in period 1 and single in period 2 is identical and we omit it for brevity.)

As  $\mu$  is dynamically stable,  $\mu(m_1) = m_1w_1 \succ_{m_1} m_1m_1$ . Since  $\succ_{m_1} \in \bar{\mathcal{S}}_{m_1}$ ,

$$w_1w_1 \succ_{m_1} \mu(m_1) = m_1w_1 \succ_{m_1} m_1m_1.$$

Furthermore,  $\mu(w_1) \succ_{w_1} m_1m_1$  as otherwise  $m_1$  and  $w_1$  could block  $\mu$ . As  $\mu(w_1) \succ_{w_1} w_1w_1$  and  $\succ_{w_1} \in \bar{\mathcal{S}}_{w_1}$ , there exists  $m_2 \in M$ , such that

$$m_2m_2 \succ_{w_1} \mu(w_1) = m_2m_1 \succ_{w_1} m_1m_1.$$

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<sup>27</sup>This case may occur if  $\mu_{t-1}(m) = m$ .

As above,  $\mu(m_2) \succ_{m_2} w_1 w_1$  as otherwise  $m_2$  and  $w_1$  could block  $\mu$ . As  $\mu(m_2) \succ_{m_2} m_2 m_2$  and  $\succ_{m_2} \in \bar{\mathcal{S}}_{m_2}$ , there exists  $w_2 \in W$  such that

$$w_2 w_2 \succ_{m_2} \mu(m_2) = w_1 w_2 \succ_{m_2} w_1 w_1.$$

Continuing by induction, suppose that for  $k \geq 2$  there exists distinct men  $m_2, \dots, m_k$  and distinct women  $w_2, \dots, w_k$  such that for each  $k' \leq k$

$$m_{k'} m_{k'} \succ_{w_{k'-1}} \mu(w_{k'-1}) = m_{k'} m_{k'-1} \succ_{w_{k'-1}} m_{k'-1} m_{k'-1}. \quad (\text{A.1})$$

and

$$w_{k'} w_{k'} \succ_{m_{k'}} \mu(m_{k'}) = w_{k'-1} w_{k'} \succ_{m_{k'}} w_{k'-1} w_{k'-1}. \quad (\text{A.2})$$

We will show that we can find a new  $m_{k+1}$  and a new  $w_{k+1}$  satisfying (A.1) and (A.2).

Given (A.2) it follows that  $\mu(w_k) \succ_{w_k} m_k m_k$  and  $\mu_2(w_k) = m_k$ . Thus, there exists  $m_{k+1} \in M$  such that

$$m_{k+1} m_{k+1} \succ_{w_k} \mu(w_k) = m_{k+1} m_k \succ_{w_k} m_k m_k.$$

As  $m_{k+1}$  and  $w_k$  cannot block  $\mu$ ,  $\mu(m_{k+1}) \succ_{m_{k+1}} w_k w_k$ . Clearly  $m_{k+1} \neq m_1, \dots, m_k$ . Otherwise  $\mu(m_{k+1}) = w_k i = \mu(m_{k'}) = w_{k'-1} w_{k'}$ , which implies  $w_{k'-1} = w_k$ . But this contradicts the assumption that  $w_1, \dots, w_k$  are distinct.

It follow, therefore, that there exists  $w_{k+1} \in W$  such that

$$w_{k+1} w_{k+1} \succ_{m_{k+1}} \mu(m_{k+1}) = w_k w_{k+1} \succ_{m_{k+1}} w_k w_k.$$

Clearly,  $w_{k+1} \neq w_k$ . If instead  $w_{k+1} = w_{k'}$  for  $k' < k$ , then  $\mu(w_{k+1}) = \mu(w_{k'}) = m_{k'+1} m_{k'}$  implying that  $m_{k+1} = m_{k'}$ , which contradicts the preceding analysis. Therefore  $w_{k+1}$  is distinct from  $w_1, \dots, w_k$ .

Thus, continuing in this manner, we can construct a sequence of distinct men  $(m_1, \dots)$ , and an sequence of distinct women  $(w_1, \dots)$ , satisfying (A.1) and (A.2). However, this is impossible since there is a finite number of men and women in the market.  $\square$

**Proof of Lemma 3.** By definition,  $i P_{\succ_i} \mu_t(i) \iff \bar{i} \succ_i \mu(i)$ . Thus,  $\mu$  can be blocked by agent  $i$  if and only if  $\mu_t$  can be blocked by  $i$ . If  $\mu$  can be blocked by  $m$  and  $w$ , these agents can block it in period 1 (Lemma 2). Thus,  $m$  and  $w$  can block  $\mu$  if and only if they can block  $\mu_t$ . Thus,  $\mu$  is stable if and only if  $\mu_t$  is stable.  $\square$

**Proof of Lemma 4.** As  $\mu \in \mathbb{D}$ ,  $\mu(i) \succsim_i \bar{i}$ .  $\succsim_i \in \mathcal{A} \cap \bar{\mathcal{S}}_i$  implies  $\overline{\mu_t(i)} \succsim_i \bar{i}$  for every  $t$ . Thus,  $\mu_t(i) R_{\succsim_i} i$ .  $\square$

**Proof of Lemma 5.** By Lemma 3 it is sufficient to verify that  $\overline{\mu_T} \in \mathbb{D}$ . Noting Lemmas A.2 and 2 we need only check a few cases. Fix  $i$  and let  $t$  be the smallest index such that  $\mu_t(i) = \dots = \mu_T(i)$ . We may further assume that  $i \neq \mu_t(i)$  as otherwise the conclusion is trivial. Let  $\hat{\mu}(i) = (\mu_{<t}, \bar{i}_{\geq t})$ . By construction,  $\hat{\mu}(i) \supseteq \mu(i)$  and  $\mu(i) \succsim_i \hat{\mu}(i)$ . As  $\succsim_i \in \bar{\mathcal{S}}_i$ ,  $\exists \succ'_i \in \mathcal{S}_i$  such that  $\succsim_i \in \Upsilon(\succ'_i)$ . Hence,  $\mu(i) \succ'_i \hat{\mu}(i)$ . As  $\mu(i)$  and  $\hat{\mu}(i)$  may differ only in periods  $t, t+1, \dots, T$ , and  $\mu_t(i) = \dots = \mu_T(i)$ ,  $\mu_T(i) R_{\succ'_i} i$ , which implies  $\mu_T(i) R_{\succ_i} i$ . Thus,  $i$  cannot block  $\overline{\mu_T}$ .

Choose any pair  $m$  and  $w$ . Assume that  $\mu_T(m) \neq w$  (otherwise, the argument is trivial). As  $\mu \in \mathbb{D}$ ,  $m$  and  $w$  cannot block it. Thus, without loss of generality,  $\mu(m) \succ_m (\mu_{<t}(m), \bar{w}_{\geq t})$  where  $t$  is the smallest index such that  $\mu_t(m) = \dots = \mu_T(m)$ . As above,  $\succ_m \in \bar{\mathcal{S}}_m$  implies that  $\mu_T(m) P_{\succ_m} w$ . Hence,  $m$  is unwilling to block  $\overline{\mu_T}$  with  $w$ . Therefore,  $\overline{\mu_T} \in \mathbb{D}$ .  $\square$

**Proof of Lemma 6.** By Lemma 4,  $\mu_t(i) R_{\succ_i} i$  for all  $i$ . Thus,  $\overline{\mu_t}(i) \succsim_i \bar{i}$  and  $i$  cannot block  $\mu'$  alone. Instead, suppose that  $m$  and  $w$  can block  $\overline{\mu_t}$ . By Lemma 2,  $m$  and  $w$  can period-1 block  $\overline{\mu_t}$ . Thus,  $w P_{\succ_m} \overline{\mu_t}(m)$  and  $m P_{\succ_w} \overline{\mu_t}(w)$ . As  $\mu \in \mathbb{D}$ ,  $m$  and  $w$  cannot period-1 block  $\mu$ . Thus,  $\mu(m) \succ_m \bar{w}$  or  $\mu(w) \succ_w \bar{m}$ . Without loss of generality, suppose  $\mu(m) \succ_m \bar{w}$ . Since  $\succ_m \in \mathcal{A}_m$ ,  $\mu_t(m) R_{\succ_m} w$ . But then  $\overline{\mu_t}(m) R_{\succ_m} w P_{\succ_m} \overline{\mu_t}(m)$ , which is a contradiction. Thus,  $\overline{\mu_t} \in \mathbb{D}$ .  $\square$

**Proof of Lemma 7.** To prove part (i) we argue by contradiction. Without loss of generality, suppose  $\mu(m_1)$  is volatile and  $\overline{\mu_t(m_1)} \succ_{m_1} \mu(m_1)$  for all  $t$ . As  $\mu \in \mathbb{D}$ ,  $\mu(m_1) \succ_{m_1} \overline{m_1}$ . Thus,  $\mu_t(m_1) = w_1 \in W$  for some  $t$  and  $\bar{w}_1 \succ_{m_1} \mu(m_1)$ .  $\mu(w_1) \succ_{w_1} \overline{m_1}$ ; else,  $w_1$  and  $m_1$  can block  $\mu$ .

As  $m_1$  is not matched to  $w_1$  for all  $t$ ,  $\mu(w_1)$  is volatile. By Lemma A.1,  $\exists m_2 \in M$  such that

$$\overline{m_2} \succ_{w_1} \mu(w_1) \succ_{w_1} \overline{m_1}$$

and  $\mu_{t(w_1)}(w_1) = m_2$  for some period  $t(w_1)$ . Clearly,  $m_1 \neq m_2$ .

As  $\mu \in \mathbb{D}$ ,  $\mu(m_2) \succ_{m_2} \overline{w_1}$ ; else,  $m_2$  and  $w_1$  could block  $\mu$ . Since  $m_2$  is not matched to  $w_1$

for all  $t$ ,  $\mu(m_2)$  is volatile. By Lemma A.1,  $\exists w_2 \in W$  such that

$$\overline{w_2} \succ_{m_2} \mu(m_2) \succ_{m_2} \overline{w_1}$$

and  $\mu_{t(m_2)}(m_2) = w_2$  for some period  $t(m_2)$ . Clearly  $w_2 \neq w_1$ .

We continue by induction. Suppose that for all  $2 \leq k' \leq k$ :

1.  $\overline{m_{k'}} \succ_{w_{k'-1}} \mu(w_{k'-1}) \succ_{w_{k'-1}} \overline{m_{k'-1}}$ ;
2.  $\overline{w_{k'}} \succ_{m_{k'}} \mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}}$ ;
3.  $\mu_{t(w_{k'-1})}(w_{k'-1}) = m_{k'}$ ;
4.  $\mu_{t(m_{k'})}(m_{k'}) = w_{k'}$ ; and,
5.  $m_1, \dots, m_k$  and  $w_1, \dots, w_k$  are all distinct.

We will show that we can find  $m_{k+1}$  and  $w_{k+1}$  distinct from those already identified satisfying 1–5.

First, consider  $w_k$ . As  $\mu \in \mathbb{D}$ ,  $\mu(w_k) \succ_{w_k} \overline{m_k}$ ; else,  $w_k$  and  $m_k$  can block  $\mu$ . By Lemma A.1,  $\exists m_{k+1} \in M$  such that

$$\overline{m_{k+1}} \succ_{w_k} \mu(w_k) \succ_{w_k} \overline{m_k}$$

and  $\mu_{t(w_k)}(w_k) = m_{k+1}$  for some period  $t(w_k)$ .

Clearly  $m_{k+1} \neq m_k$ . If  $m_{k+1} = m_1$ , then  $w_k$  and  $m_1$  can block  $\mu$ , which is a contradiction.

Finally, if  $m_{k+1} = m_{k'}$  for  $1 < k' < k$ , we know that

$$\overline{w_{k'}} \succ_{m_{k'}} \mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}}$$

and  $\mu_{t(w_{k'-1})}(w_{k'-1}) = m_{k'}$ . As  $\mu \in \mathbb{D}$ ,  $\mu(m_{k'}) \succ_{m_{k'}} \overline{w_k}$ . Since  $\succ_{m_{k'}} \in \mathcal{A}_{m_{k'}}$  and  $w_{k'-1}$  and  $w_k$  are distinct members encountered during the plan  $\mu(m_{k'})$ ,

$$\mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}} \implies \overline{w_k} \succ_{m_{k'}} \overline{w_{k'-1}}$$

and

$$\mu(m_{k'}) \succ_{m_{k'}} \overline{w_k} \implies \overline{w_{k'-1}} \succ_{m_{k'}} \overline{w_k}.$$

Clearly, this is a contradiction. Therefore  $m_{k+1} \neq m_{k'}$ . Hence,  $m_{k+1}$  is distinct from each  $m_1, \dots, m_k$ .

Now consider  $m_{k+1}$ . As  $\mu \in \mathbb{D}$ ,  $\mu(m_{k+1}) \succ_{m_{k+1}} \overline{w_k}$ ; else  $m_{k+1}$  and  $w_k$  can block  $\mu$ . By Lemma A.1,  $\exists w_{k+1} \in W$  such that

$$\overline{w_{k+1}} \succ_{m_{k+1}} \mu(m_{k+1}) \succ_{m_{k+1}} \overline{w_k}.$$

and  $w_{k+1} = \mu_{t(m_{k+1})}(m_{k+1})$  for some period  $t(m_{k+1})$ .

Clearly,  $w_{k+1} \neq w_k$ . Suppose  $w_{k+1} = w_{k'}$  for some  $k' < k$ . We know that

$$\overline{m_{k'+1}} \succ_{w_{k'}} \mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k'}},$$

$\mu_{t(m_{k'})}(w_{k'}) = m_{k'}$  and  $\mu_{t(m_{k+1})}(w_{k'}) = m_{k+1}$ . As  $\mu \in \mathbb{D}$ ,  $\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k+1}}$ ; else,  $m_{k+1}$  and  $w_{k'} = w_{k+1}$  can block  $\mu$ . Since  $\succ_{w_{k'}} \in \mathcal{A}_{w_{k'}}$  and  $m_{k'}$  and  $m_{k+1}$  are distinct

$$\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k'}} \implies \overline{m_{k+1}} \succ_{w_{k'}} \overline{m_{k'}}$$

and

$$\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k+1}} \implies \overline{m_{k'}} \succ_{w_{k'}} \overline{m_{k+1}}$$

Clearly, this is a contradiction. Therefore,  $w_{k+1} \neq w_{k'}$  for all  $k' \leq k$ .

Proceeding by induction we can construct an infinite sequence of men  $m_2, m_3, \dots$  and women  $w_2, w_3, \dots$  satisfying conditions 1–5 above. However, this is not possible as there is a finite number of agents. Thus, if  $\mu(m_1)$  is volatile, there exists a period  $t$  such that  $\mu(m_1) \succ_{m_1} \overline{\mu_t(m_1)}$ .

To verify point (ii) it is sufficient to observe that when  $\mu_t(i) \neq \mu_{t'}(i)$ ,  $\mu(i) \succ_i \overline{\mu_t(i)} \implies \overline{\mu_{t'}(i)} \succ_i \overline{\mu_t(i)}$  and  $\mu(i) \succ_i \overline{\mu_{t'}(i)} \implies \overline{\mu_t(i)} \succ_i \overline{\mu_{t'}(i)}$ , which is a contradiction as  $\succ_i \in \mathcal{A}_i$ .  $\square$

**Proof of Lemma 8.** Consider agent  $i$ . If  $\mu(i) = \mu'(i)$ , the lemma's first part is clearly true. Suppose  $\mu(i) \succ_i \mu'(i)$ . As both matchings are maximally persistent,  $\mu(i) = \overline{\mu_t(i)}$  and  $\mu'(i) = \overline{\mu_{t'}(i)}$  for all  $t$  and  $t'$ . Thus,  $\overline{\mu_t(i)} \succ_i \overline{\mu_{t'}(i)}$ . Therefore,  $\mu(i) \succ_i^* \mu'(i)$ . Conversely, if  $\mu(i) \succ_i^* \mu'(i)$ , then  $\mu(i) \succ_i \mu'(i)$  follows immediately from the definition of  $\succ_i^*$ .

Now consider the collective preference. If  $\mu = \mu'$ , the lemma's conclusion is again trivial. If  $\mu \succ_M \mu'$ , then  $\mu(m) \succ_m \mu'(m)$  for all  $m$  and  $\mu(m') \succ_m \mu'(m')$  for some  $m'$ . As  $\mu$  and  $\mu'$  are maximally-persistent matchings,  $\mu(m) \succ_m^* \mu'(m)$  for all  $m$  and  $\mu(m') \succ_m^* \mu'(m')$  for some  $m'$ . Hence,  $\mu \succ_M^* \mu'$ . Conversely,  $\mu \succ_M^* \mu' \implies \mu(m) \succ_m^* \mu'(m) \forall m \in M \implies \mu(m) \succ_m \mu'(m) \forall m \in M \implies \mu \succ_M \mu'$ .  $\square$

**Proof of Lemma 9.** Let  $\mu'$  be the  $\succsim_M$ -maximal matching in  $\mathbb{P}$ . Since  $(\mathbb{P}, \succsim_M)$  is a finite lattice, this matching exists and is unique. Let  $\mu \in \mathbb{D}$  and suppose  $\mu' \not\prec_M^* \mu$ . Hence,  $\exists m \in M$  such that  $\mu'(m) \not\prec_m^* \mu(m)$ . This implies  $\mu(m)$  is volatile. Thus, there exists a period  $t'$  such that  $\mu(m) \succ_m \overline{\mu_{t'}(m)}$  and for all  $t \neq t'$ ,  $\overline{\mu_t(m)} \succ_m \mu(m)$ . However,  $\overline{\mu_t} \in \mathbb{P}$  for all  $t$ . Thus,  $\mu'(m) \succsim_m \overline{\mu_t(m)} \succ_m \mu(m) \succ_m \overline{\mu_{t'}(m)}$ . But this implies  $\mu'(m) \succ_m^* \mu(m)$ , which is a contradiction.  $\square$

**Proof of Lemma 10.** Suppose  $\mu \succ_M^* \mu'$ . Thus, there exists at least one man and at least one woman for whom  $\mu(i) \neq \mu'(i)$ . If  $\mu' \not\prec_W^* \mu$ , then  $\mu'(w) \not\prec_w^* \mu(w)$  for some  $w \in W$ . Thus,  $\mu'(w) \neq \mu(w)$  and  $\mu'(w) \not\prec_w^* \mu(w)$ . There are two cases.

1. There exist  $t$  and  $t'$  such that  $\overline{\mu_t(w)} \succ_w \overline{\mu_{t'}(w)}$ . Hence, there exists  $m \in M$  such that  $m = \mu_t(w)$  and  $\bar{m} = \overline{\mu_t(w)} \succ_w \overline{\mu_{t'}(w)}$ . Since  $\mu \succ_M^* \mu'$ , it follows that for this agent  $m$ ,  $\mu(m) \succ_m^* \mu'(m)$  and for  $t$  and  $t'$ ,  $\bar{w} = \overline{\mu_t(m)} \succ_m \overline{\mu_{t'}(m)}$ . As  $\mu_t(w) = m \neq \mu_{t'}(w)$ , it must be the case that  $\overline{\mu_t(m)} \succ_m \overline{\mu_{t'}(m)}$ .

By Lemma 6,  $\overline{\mu_{t'}} \in \mathbb{D}$ . However,  $\overline{\mu_{t'}(w)} = \overline{\mu_{t'}(w)}$  and  $\overline{\mu_{t'}(m)} = \overline{\mu_{t'}(m)}$ . Thus,  $m$  and  $w$  can period-1 block  $\overline{\mu_{t'}}$ , which is a contradiction.

2. Suppose  $\mu(w) \succ_w \mu'(w)$ . By Lemma A.1, there exist  $t$  and  $t'$  such that  $\overline{\mu_t(w)} \succ_w \mu(w)$  and  $\mu'(w) \succ_w \overline{\mu_{t'}(w)}$ . Thus, the same argument as case (1) applies.

As neither case applies, we arrive at a contradiction. Hence,  $\mu' \succ_W^* \mu$ .  $\square$

**Proof of Theorem 5.** Suppose  $i$  can block  $\mu_{>t}^*$ , i.e.

$$(\mu_{(t,t')}^*(i), \bar{i}_{\geq t'}) \succ_i^{\mu_{\leq t}^*(i)} \mu_{>t}^*(i) \iff (\mu_{\leq t}^*(i), \mu_{(t,t')}^*(i), \bar{i}_{\geq t'}) \succ_i (\mu_{\leq t}^*(i), \mu_{>t}^*(i)).$$

Thus,  $i$  can block  $\mu^*$ , which is a contradiction.

If  $\{m, w\}$  can block  $\mu_{>t}^*$ , then

$$(\mu_{(t,t')}^*(i), \mu_{\geq t'}^{\{m,w\}}(i)) \succ_i^{\mu_{\leq t}^*(i)} \mu_{>t}^*(i) \iff (\mu_{\leq t}^*(i), \mu_{(t,t')}^*(i), \mu_{\geq t'}^{\{m,w\}}(i)) \succ_i (\mu_{\leq t}^*(i), \mu_{>t}^*(i))$$

for  $i \in \{m, w\}$ . Thus,  $\{m, w\}$  can block  $\mu^*$ , which is a contradiction. Hence,  $\mu_{>t}^*$  is dynamically stable.  $\square$

**Proof of Lemma 11.**

1. We propose a lexicographic extension of  $\succsim_i \in \mathcal{S}_i$ . For all  $x = x_1 \cdots x_T$  and  $y = y_1 \cdots y_T$ , define  $\hat{\succsim}_i$  as follows:

$$x_1 \cdots x_T j \hat{\succsim}_i y_1 \cdots y_T k \iff \begin{cases} x \succsim_i y \\ \text{or} \\ x = y \ \& \ j P_{\succsim_i} k \end{cases} \quad (\text{A.3})$$

It is simple to verify that  $\hat{\succsim}_i$  reflects  $P_{\succsim_i} = P_{\hat{\succsim}_i}$ . Thus,  $\hat{\succsim}_i \in \hat{\mathcal{S}}_i$ .

2. Given  $\succsim_i \in \bar{\mathcal{S}}_i$ , let  $\succsim'_i \in \mathcal{S}_i$  be such that  $\succsim_i \in \Upsilon(\succsim'_i)$ . Let  $\hat{\succsim}'_i \in \hat{\mathcal{S}}_i$  be the one-period lexicographic extension, as in (A.3), of  $\succsim'_i$ .  $\hat{\succsim}'_i$  reflects  $P_{\hat{\succsim}'_i} = P_{\succsim'_i}$ . Define  $\hat{\succsim}_i$  as follows:

- (a) If  $\hat{x}_T \neq \hat{x}_{T+1}$  or  $\hat{y}_T \neq \hat{y}_{T+1}$ , then  $\hat{x} \hat{\succsim}'_i \hat{y} \iff \hat{x} \hat{\succsim}_i \hat{y}$ .
- (b) If  $\hat{x}_T = \hat{x}_{T+1}$  and  $\hat{y}_T = \hat{y}_{T+1}$ , then  $\hat{x}_{\leq T} \succsim_i \hat{y}_{\leq T} \iff \hat{x} \hat{\succsim}_i \hat{y}$ .

$\hat{\succsim}_i$  is a one-period extension of  $\succsim_i$ . To confirm that  $\hat{\succsim}_i \in \Upsilon(\hat{\succsim}'_i)$ , we check two conditions.

- (a) Suppose  $\hat{x} \hat{\succsim}'_i \hat{y}$  and  $\hat{x} \not\supseteq \hat{y}$ . To work toward a contradiction, assume  $\hat{y} \hat{\succsim}_i \hat{x}$ . Then  $\hat{y}_T = \hat{y}_{T+1}$ . Since  $\hat{x} \not\supseteq \hat{y}$  then  $\hat{x}_T = \hat{x}_{T+1}$ . Therefore,  $\hat{y}_{\leq T} \succsim_i \hat{x}_{\leq T}$  as well. But,  $\hat{x} \not\supseteq \hat{y}$  also implies  $\hat{x}_{\leq T} \supseteq \hat{y}_{\leq T}$  and, therefore,  $\hat{y}_{\leq T} \succsim'_i \hat{x}_{\leq T}$ . But then  $\hat{y} \hat{\succsim}'_i \hat{x}$ , which is a contradiction. Therefore,  $\hat{x} \hat{\succsim}_i \hat{y}$ .
- (b) Suppose  $\hat{x} \parallel \hat{y}$ . This implies  $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$ ; else, the two partnership plans could be ordered by  $\supseteq$ .

First, suppose  $\hat{x} \hat{\succsim}'_i \hat{y}$ . As  $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$ , it follows that  $\hat{x}_{\leq T} \succsim'_i \hat{y}_{\leq T}$ . Suppose  $\hat{y} \hat{\succsim}_i \hat{x}$ . This is possible only if  $\hat{y}_T = \hat{y}_{T+1}$  and  $\hat{y}_{\leq T} \succsim_i \hat{x}_{\leq T}$ . But if  $\hat{x}_{\leq T} \succsim'_i \hat{y}_{\leq T}$ , then it must be that  $\hat{y}_{\leq T} \supseteq \hat{x}_{\leq T}$  as  $\succsim_i \in \Upsilon(\succsim'_i)$ . This implies  $(\hat{y}_{\leq T}, \hat{y}_{T+1}) \supseteq (\hat{x}_{\leq T}, \hat{x}_{T+1})$ , which contradicts  $\hat{y} \not\supseteq \hat{x}$ . Thus,  $\hat{x} \hat{\succsim}_i \hat{y}$ .

Conversely, let  $\hat{x} \hat{\succsim}_i \hat{y}$ . To derive a contradiction, suppose  $\hat{y} \hat{\succsim}'_i \hat{x}$ . Thus,  $\hat{x}_T = \hat{x}_{T+1}$  and  $\hat{x}_{\leq T} \succsim_i \hat{y}_{\leq T}$ . However, since  $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$ ,  $\hat{y}_{\leq T} \succsim'_i \hat{x}_{\leq T}$ . Thus,  $\hat{x}_{\leq T} \supseteq \hat{y}_{\leq T}$ . As  $\hat{x}_T = \hat{x}_{T+1}$ , then  $\hat{x} \supseteq \hat{y}$ —a contradiction. Therefore,  $\hat{x} \hat{\succsim}'_i \hat{y}$ .

□

**Proof of Theorem 6.** Let  $\succsim'_i \in \mathcal{S}_i$  be such that  $\succsim_i \in \Upsilon(\succsim'_i)$ . Define  $\hat{\succsim}'_i$  analogously, i.e.



$\succ_i \in \Upsilon(\hat{\succ}_i')$ . Since  $\mu^*$  is dynamically stable,  $i$  cannot block it. As  $\hat{\succ}_i$  extends  $\succ_i$ ,

$$\mu^*(i) \succ_i (\mu_{<t}^*(i), \bar{i}_{[t,T]}) \implies \underbrace{(\mu_1^*(i), \dots, \mu_T^*(i), \mu_T^*(i))}_{\hat{\mu}^*(i)} \hat{\succ}_i \underbrace{(\mu_{<t}^*(i), \bar{i}_{[t,T]}, i)}_{(\hat{\mu}_{<t}^*(i), \bar{i}_{[t,T]})}.$$

Hence,  $i$  cannot block  $\hat{\mu}^*$  in period  $t \leq T$ .

Suppose  $i$  can block  $\hat{\mu}^*$  in period  $\hat{T}$ , i.e.  $(\hat{\mu}_{\leq T}^*(i), i) \hat{\succ}_i \hat{\mu}^*(i)$ . Thus,  $\hat{\mu}_T^*(i) \neq i$  and, therefore,  $\hat{\mu}^*(i) \supseteq (\hat{\mu}_{\leq T}^*(i), i)$ . This implies  $(\hat{\mu}_{\leq T}^*(i), i) \hat{\succ}_i' \hat{\mu}^*(i)$  and, therefore,  $iP_{\hat{\succ}_i'} \hat{\mu}_T^*(i) \implies iP_{\hat{\succ}_i'} \hat{\mu}_T^*(i)$ . Let  $t' \leq T$  be the smallest value such that  $\mu_{t'}^*(i) = \dots = \mu_T^*(i)$ . Thus,

$$\tilde{\mu}(i) \equiv (\mu_{<t'}^*(i), \bar{i}_{\geq t'}) \succ_i' (\mu_{<t'}^*(i), \mu_{t'}^*(i), \dots, \mu_T^*(i)) = \mu^*(i).$$

But,  $\tilde{\mu}(i) \supseteq \mu^*(i)$  and thus,  $\tilde{\mu}(i) \succ_i \mu^*(i)$ . Hence,  $\mu^*$  can be period- $t'$  blocked by  $i$  in the  $T$ -period economy. This is a contradiction. Therefore,  $i$  cannot block  $\hat{\mu}^*$ .

Consider the pair  $m$  and  $w$ . They cannot block  $\mu^*$  in period  $t \leq T$ . From the definition of stability,  $\mu^*(m) \succ_m (\mu_{<t}^*(m), \bar{w}_{[t,T]})$  and  $\mu^*(w) \succ_w (\mu_{<t}^*(w), \bar{m}_{[t,T]})$ . Hence,

$$\underbrace{(\mu_{\leq T}^*(m), \mu_T^*(m))}_{\hat{\mu}^*(m)} \hat{\succ}_m (\mu_{<t}^*(m), \bar{w}_{[t,T]}, w) \quad \text{and} \quad \underbrace{(\mu_{\leq T}^*(w), m)}_{\hat{\mu}^*(w)} \hat{\succ}_w (\mu_{<t}^*(w), \bar{m}_{[t,T]}, m).$$

Thus,  $m$  and  $w$  cannot block  $\hat{\mu}^*$  in period- $t$ .

Suppose  $m$  and  $w$  can period- $\hat{T}$  block  $\hat{\mu}^*$ :

$$\tilde{\mu}(m) \equiv (\hat{\mu}_{\leq T}^*(m), w) \hat{\succ}_m \hat{\mu}^*(m) \quad \text{and} \quad \tilde{\mu}(w) \equiv (\hat{\mu}_{\leq T}^*(w), m) \hat{\succ}_w \hat{\mu}^*(w).$$

Of course, this implies  $\hat{\mu}_{T+1}^*(m) = \mu_T^*(m) \neq w$ . By reasoning analogous to the single-agent case above,  $\hat{\mu}^*(m) \supseteq \tilde{\mu}(m) \implies \tilde{\mu}(m) \hat{\succ}_m' \hat{\mu}^*(m) \iff wP_{\hat{\succ}_m'} \hat{\mu}_T^*(m)$ . Similarly,  $mP_{\hat{\succ}_w'} \mu_T^*(w)$ .

Let  $t'_i$  be the smallest index such that  $\mu_{t'_i}^*(i) = \dots = \mu_T^*(i)$  and let  $t' = \max\{t'_m, t'_w\}$ . Then,

$$(\mu_{<t'}^*(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m' (\mu_{<t'}^*(m), \mu_{[t',T]}^*(m), w) = \tilde{\mu}(m).$$

Given the definition of  $t'$ ,  $(\mu_{<t'}^*(m), \mu_{[t',T]}^*(m), w) \not\triangleright (\mu_{<t'}^*(m), \bar{w}_{[t',T]}, w)$ . And so,

$$(\mu_{<t'}^*(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m \tilde{\mu}(m).$$

But then  $(\mu_{<t'}^*(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m \hat{\mu}^*(m)$  and likewise for  $w$ ,  $(\mu_{<t'}^*(w), \bar{w}_{[t',T]}, w) \hat{\succ}_w \hat{\mu}^*(w)$ . Thus,  $m$  and  $w$  can block  $\hat{\mu}^*$  in period  $t' \leq T$ . Above, we established that this is not possible.  $\square$

