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Core and Periphery in Endogenous Networks

Daniel Hojman
John F. Kennedy School of Government
Harvard University

Adam Szeidl
University of California - Berkeley

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Daniel Hojman
Harvard University

Adam Szeidl
UC - Berkeley

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Abstract

Many economic and social networks share two common organizing features: (1) a core-periphery structure; (2) positive correlation between network centrality and payoffs. In this paper, we build a model of network formation where these features emerge endogenously. In our model, the unique equilibrium network architecture is a periphery-sponsored star. In this equilibrium, one player, the center, maintains no links and achieves a high payoff, while all other players maintain a single link to the center and achieve lower payoffs. With heterogeneous groups, equilibrium networks are interconnected stars. We show that small minorities tend to integrate while large minorities are self-sufficient. Although any player can be the center in a static equilibrium, evolution selects the agent with most valuable resources as the center in the long run. In particular, even small inequalities in resources can lead to large payoff inequality because of the endogenous social structure. Our main results are robust to the introduction of transfers and bargaining over link costs.

JEL Classification: A14, C72, C73

Keywords: network formation, centrality, evolution, social networks, social capital

*E-mail: Daniel_Hojman@ksg.harvard.edu and szeidl@econ.berkeley.edu. We are grateful to Attila Ambrus, Ed Glaeser, Sanjeev Goyal, Kenneth Judd, Larry Katz, Markus Mobius, Nolan Miller, Jesse Shapiro, Jeremy Tobacman, Richard Zeckhauser, numerous seminar participants, and especially Drew Fudenberg and Jerry Green for helpful comments and suggestions.

"The social capital metaphor is that the people who do better are somehow better connected. Certain people or certain groups are connected to certain others, trusting certain others, obligated to support certain others, dependent on the exchange with certain others. Holding a better position in the structure of these exchanges can be an asset in its own right. This asset is social capital"

Ronald S. Burt, 2005

1 Introduction

Networks influence behavior in many economic contexts, and network structure can play an important role in determining agents' access to various resources.[1] Remarkably, a wide variety of economic and social networks share two common organizing features. First, these networks exhibit a *core-periphery structure*. A small number of central agents or "hubs" agglomerate a disproportionate amount of connections, while most other agents maintain few relationships. Hubs channel information and rumors, broker knowledge between otherwise disconnected agents, and bridge other resources between agents in the periphery. Examples of this core-periphery structure include networks of individuals, such as networks of scientific collaboration (Moody, 2004; Newman, 2004; Goyal et al. 2005), friendship (Adaic and Adam, 2003), community support (Tardy and Hale, 1998), informal networks of advice in the workplace (Krackhardt and Hansen, 1993, Cross et al., 2001), as well as interlocking directorates of corporations (Mizruchi, 1996). Similar core-periphery structures have been observed at the level of organizations, like the R&D alliances in knowledge-intensive industries (Powell et al., 1996; Gulati and Gargiulo, 1999) and international trade (Snyder and Kick, 1979). Barabasi and Albert (2003) summarize evidence on the emergence of hubs in many large networks.

A second empirical regularity is the positive correlation between network position and various measures of performance and payoffs. A body of evidence shows that central agents do better than peripheral agents in many environments. Burt (2000) surveys the literature in sociology on the performance of well-connected individuals, and shows that in managerial networks promotion rates and high evaluation are positively related to centrality. Cross and Cummings (2004) argue that the higher performance of central agents in an organization's communication network is due to better access to non-redundant information. Balwin et al. (1997) show that in a sample of MBA students,

[1] Connnections are a leading source of information about employment opportunities (Granovetter, 1974; Montgomery, 1991; Topa, 2001; Calvo-Armengol and Jackson, 2004), consumption products (Katz and Lazersfeld, 1955; Reigen and Kernan, 1986), and health alternatives (Feldman and Spencer, 1965). Fund managers form portfolios based on advice from other investors (Hong et al. 2004). The adoption of new technologies in agriculture is strongly affected by learning spillovers (Foster and Rosenzweig, 1995; Conley and Udry, 2004). R&D alliances are important in the development of successful innovations (Powell et al, 1996). Network effects play an important role in determining attitudes toward welfare (Bertrand et al. 2000), engagement in criminal behavior (Glaeser et al., 1996) and the accumulation of human capital (Katz et al., 2001).

centrality is positively related to academic performance. Similarly, in labor markets, centrality is associated with finding better paid jobs (Lin 2001). Across organizations, Powell et al (1999) and Ahuja (2000) show that centrality is positively related to volume of patenting, earnings, and higher chances of survival.[2]

This paper builds a theory of endogenous network formation that leads to (1) the emergence of core-periphery structures, and (2) a positive relationship between network position and payoffs. Our model is based on network externalities: agents maintain links both to obtain direct benefits, and also to access indirect benefits via the networks of the people they link to. We show that when maintaining links is costly, such externalities lead to agglomeration benefits, the emergence of hubs and an associated payoff inequality.

In Section 2, we present the basic network formation model. The key assumptions of the model are that network benefits exhibit strongly decreasing returns to scale, and that there are frictions in communication. We provide microfoundations for these assumptions using two stylized examples. In the first example, agents seek a partner with whom to complete a task by asking their close neighbors in the network, who in turn can refer to their neighbors, and so on. In the second example, agents have imperfect information about the state of the world, but can obtain noisy signals about the information of others through the social network. In both examples, network externalities exhibit decreasing returns at the individual level: with a large neighborhood, access to additional people has little effect on the likelihood of finding a partner, or on the precision of the estimate. Frictions in communication imply that benefits decrease with the distance between players; we require these benefits to vanish beyond some finite threshold distance.

Our first main result is presented in Section 3. We show that in the network formation model explained above, the unique Nash equilibrium network architecture is the *periphery-sponsored star*. Figure 1 demonstrates the shape of the equilibrium network. In words, there is a single player, the center, who maintains no links, and all other players maintain one link to the center. It is easy to see that the periphery-sponsored star is an equilibrium of the model. Establishing that there are no other equilibrium architectures is more challenging. To see the logic behind this result, first note that limits to communication induce agents to form links that keep them close to one another. As a result, in any equilibrium there must exist agents who have a large number of close connections. In such a network, due to strong decreasing returns, all agents find it optimal to maintain at most one link: connecting to a well-connected individual provides access to a high payoff, which makes forming a second link suboptimal.

This characterization of equilibrium shows that our model can explain the endogenous emergence of hubs: any equilibrium network has a well-defined center. Importantly, the equilibrium architecture is unique, which is consistent with the core-periphery structure being so pervasive in

[2] While there is no clear evidence on causality, researchers often argue that having better connections can in itself lead to higher payoffs.

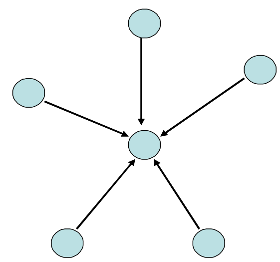

Figure 1: Periphery-sponsored star.

a variety of networks. The model also generates the positive correlation between network position and payoffs documented in the data. The equilibrium payoff of the center is higher that the payoff of peripheral agents for two distinct reasons. First, the center is closer to all agents in the economy in terms of network distance, which increases her payoff in the presence of communication frictions. Second, the center maintains no links, while peripheral agents all maintain a single costly connection. Interestingly, this payoff inequality can make the equilibrium network inefficient because the benefit function is concave.

The basic model predicts that there exists a single hub in the entire economy. In Section 4 we show that the model can also generate multiple hubs if we introduce heterogeneity across players. Formally, we assume that there are a number of groups in the population, and maintaining links within a group has a lower cost than maintaining a link across groups. Intuitively, these subcommunities may correspond to differences in geography, race, or other characteristics. In this setup, we find that any equilibrium has an "interlinked star" architecture. Subcommunities are organized into stars, which may be separate or connected to one another. This architecture appears to be a good description of co-authorship networks, where sub-disciplines within a discipline are organized in subnetworks with strong core-periphery features, which are loosely connected to each other (Newman, 2004).

We study comparative statics in the case with two subcommunities. We find that when the two groups are about the same size, separation is more likely, whereas smaller minority group is more willing to integrate. We also show that when the cost difference between the two types of connections is low, the forces of integration are so strong that a single star obtains. In contrast, when the network is very good in channeling indirect benefits, interlinked but different stars are a more likely outcome. Hence, in response to improvements in network technology, subcommunities can either "get closer or drift apart" (as in Rosenblat and Mobius, 2004), depending on whether these improvements affect the relative cost of links or the communication efficiency of the network.

In the basic model, all agents are identical. Therefore, while the equilibrium architecture is unique, the identity of the central agent is not: for every agent i in the economy there exists an equilibrium with i as the center. This result suggests that history and luck are important determinants of centrality. In Section 5 we investigate the determinants of centrality further by assuming that there exists an agent with a special quality that makes her more beneficial to link to. In this setup, there is still a multiplicity of static equilibria, and any player can be the center. However, a myopic best response dynamics with persistent randomness (as in Kandori et. al., 1993 and Young, 1993) selects a unique equilibrium in the long run, in which the center is the player with the special quality. We conclude that even though luck and history matter in the short term, qualities become more important over long horizons. Importantly, even an arbitrarily small quality advantage can lead to centrality and a significant payoff advantage. Hence, small differences in endowments can be amplified and lead to large inequality due to the endogenous social structure. Our findings are consistent with the positive empirical relationship between human capital and social capital (Helliwell and Putnam, 1999; Glaeser et al., 2002).

One limitation of our basic model is that the cost of any connection is fully borne by the agent maintaining the link. In section 6 we relax this assumption and allow agents to share link costs by means of transfers, as in Bloch and Jackson (2005). This extension is intended to capture the idea that agents may bargain to determine how their joint surplus is shared. Agents may also choose to invest in multiple links in an attempt to become central. The transfers can represent monetary payoffs, for example in a network of firms, but may also capture non-pecuniary benefits in social networks. In this context, link formation incentives are determined by the joint surplus over links. We show that the star architecture continues to be the unique equilibrium outcome if the payoff function satisfies an additional condition.[3] Loosely, we require that the incentives to form a link between two parties are increasing in the number of neighbors of either party. Importantly, the analogue of this "monotone surplus condition" always holds in the basic model without transfers. The intuitive role of the condition is to rule out fragmented networks and encourage agglomeration, leading to the star architecture.

This paper builds on and contributes to the growing literature on endogenous networks. In their influential work, Jackson and Wolinsky (1996) study a cooperative model of network formation. More closely related to our work are Bala and Goyal (2000) and Galeotti, Goyal, and Kamphorst (2005) who both study non-cooperative models of network formation. Bala and Goyal (2000) allow for a general benefit structure and show that, in the absence of frictions in communication, there are

[3]We analyze the model with transfers using two refinements of the pairwise Nash equilibrium concept introduced by Jackson and Wolinsky (1996) to study models with two-sided network formation.

a large number of pure strategy Nash equilibria, but a unique strict Nash equilibrium architecture, the center-sponsored star. In this equilibrium, the center obtains the lowest payoff in the population, in contrast with the evidence on performance and network position discussed above. Galeotti, Goyal and Kamphorst (2005) focus on a constant returns to scale benefit function with two groups. They show that for some parameters, the periphery-sponsored interlinked star can be the unique strict Nash equilibrium in the restricted case when there are very small frictions in communication. The logic is that when there are almost no frictions, a single link to any other player is sufficient to reap all benefits. In contrast, in our model, sparse networks emerge due to decreasing returns independently of the size of frictions and without the need to rely on strict Nash equilibrium. Neither of these papers discuss the selection of the center and the role of two-sided links and transfers. In addition, relative to this work, our model is more directly motivated by empirical evidence.

Other work on network formation includes Kranton and Minehart (2001) who model the endogenous formation of buyer-seller networks. Goyal and Joshi (2002) analyze the formation of R&D collaboration networks between firms in an oligopoly. Goyal and Vega-Redondo (2005) and Hojman and Szeidl (2005) study the interplay between network formation and equilibrium selection in coordination games. Jackson and Dutta (2003) and Jackson (2005) provide an overview of this growing literature.

2 A Model of Network Formation

Consider a simultaneous-move game in which N players decide on maintaining links to one another. Each link represents an economic or social connection between two players. A strategy for player i is a vector $s^i \in \{0,1\}^{N-1}$, where $s^i_j = 1$ if player i decides to link j and 0 otherwise. Any strategy profile defines an undirected graph or network denoted by $g = (s^1, ..., s^N)$ where agents are the nodes and links are the edges of this graph. Links are costly but give rise to the benefits of having direct or indirect access to other players. The cost of forming a link is $c > 0$. Links are undirected in providing access to others, that is, a link between any two players is a conduit that allows resources to flow in both directions. We allow benefits to depend on the distance in the graph between players. Formally, the distance between two players i and j is the number of edges along the shortest path between i and j in the network g. If no such path exists, the distance is set to infinity. In this model, the distance between any two players is endogenous and need not be related to a exogenous measure separation such as geographic distance or cultural differences.[4] For each player i, denote the number of links i maintains by $l^i = \sum_{j \in N} s^i_j$ and the number of players who are exactly at distance k from i in the network g by n^i_k. The payoff of player i in the network

[4]The extension in Section 4 illustrates how network distance responds endogenously to such exogenous measures of separation.

formation game is defined as

$$\pi^i(s^i, s^{-i}) = f\left(a_1 \cdot n_1^i + a_2 \cdot n_2^i + ... + a_d \cdot n_d^i\right) - c \cdot l^i, \tag{1}$$

where $f(.)$ is a strictly increasing and concave real valued benefit function. Here d is the *communication threshold*: agents who are more than d far away yield no benefit. The positive weights a_1, a_2,..., a_d measure the relative importance of neighbors at different distances. We assume that $a_1 \geq a_2 \geq ... \geq a_d$, so that more distant neighbors yield (weakly) less benefits. We also normalize $a_2 = 1$. This assumption is inconsequential, but makes it easier to state our results. Important special cases include $a_1 = a_2 = ... = a_d = 1$ in which case direct and indirect neighbors are equally important, and $a_{s+1} = \beta \cdot a_s$ with $0 < \beta < 1$, that is, geometric discounting of benefits by distance. In the sequel we will sometimes refer to the sum $\overline{n}^i = a_1 \cdot n_1^i + a_2 \cdot n_2^i + ... + a_d \cdot n_d^i$ as the *effective number of agents* or *effective neighbors* that i has access to.

By assuming that $f(.)$ is increasing and the discount weights are positive we restrict attention to the case of positive spillover from connections: The addition of an arbitrary link to the network weakly increases the benefits of all players. Allowing for congestion in the model is an interesting extension that we do not focus on in this paper.

2.1 Key assumptions

Throughout the paper, we make the following two assumptions about the payoff structure.

Assumption 1. (Limits to communication.) The communication threshold d is finite.

Assumption 1 states that distant neighbors in the network yield no benefit, which we find reasonable in many contexts. For the second assumption, we introduce the definition that $f(.)$ exhibits (M, c)*-strong decreasing returns* if it is strictly increasing, concave, and for all $m > M$ the following inequality is satisfied:

$$f(2m) - f(m) < c. \tag{2}$$

Assumption 2. (Strong decreasing returns.) The benefit function $f(.)$ exhibits (M, c)-strong decreasing returns for some $M \geq 0$.

For example when $f(.)$ is bounded, it is easy to see that for any c there exists an M such that $f(.)$ exhibits (M, c)-strong decreasing returns. Hence the commonly used exponential utility function $f(n) = -e^{-An}$ as well as power utility $f(n) = n^{1-\gamma}/(1-\gamma)$ for $\gamma > 1$ satisfy strong decreasing returns for some M. Intuitively, $f(.)$ exhibits strong decreasing returns if it grows slower than the log.[5] Note that if the weights $a_1, ..., a_d$ are proportionally changed, the definition of $f(.)$ and the threshold M in the concept of strong decreasing returns need to be modified accordingly. This is the reason for normalizing $a_2 = 1$. We discuss the implications and intuitive content of these assumptions in Section 3.

[5] Note that $f(n) = A \cdot \log(n)$ has $(0, c)$-strong decreasing returns as long as $A < c/\log 2$.

2.2 Micro-Foundations for Key Assumptions

In this section, we motivate our key assumptions by building two examples where strong decreasing returns and limits to communication arise endogenously.

2.2.1 Advice-Seeking and Knowledge Exchange in a Network

Network ties between individuals or organizations are often helpful in overcoming problems that agents would not be able to solve in isolation. For example, the developer of a project may face a problem he is not familiar with. At the same time, the developer may have skills that he does not need for the current problem, but that might be useful for other developers. Such situations can emerge in technology intensive industries, in academic research, as well as in agriculture. For example, Allen (1984) shows that engineers in R&D labs are five times as likely to use a personal source of information than an impersonal source like a scientific journal.

Formally, consider an economy where agents form costly links to one another, and then each agent i faces a problem y_i. There are K different types of problems, i.e., $y_i \in \{1, ..., K\}$, and each problem occurs with probability $1/K$. Each agent i is endowed with the ability $\alpha_i \in \{1, ..., K\}$ to solve a specific type of problem. The ex ante probability that agent i is able to solve problem y is assumed to be $1/K$ for each y. An unsolved problem yields a low payoff of -1, a solved problem gives 0. Agents can consult their neighbors to find a solution to their problem.[6] Since a solution is time sensitive, we assume that the agents can only reach neighbors within a distance of d, where $d \geq 2$. There is also a friction in the communication process: the probability of learning the ability of a neighbor at distance $\delta \in \{0, ..., d\}$ is p_δ, where $p_\delta \in (0, 1]$ is decreasing in δ. The case $p_\delta \equiv 1$ corresponds to frictionless communication.

It follows that a player with n_δ neighbors at distance δ finds no solution to her problem with probability $\prod_{\delta=1}^{d} \left[p_\delta \left(K-1\right)/K\right]^{n_\delta}$. Define a_δ implicitly by $p_\delta = \left(\frac{K-1}{K}\right)^{a_\delta - 1}$, then the agent's expected payoff takes the form

$$f(\bar{n}^i) = -\prod_{\delta=1}^{d} \left(\frac{K-1}{K}\right)^{a_\delta n_\delta} = -\left(\frac{K-1}{K}\right)^{\bar{n}^i} = -\exp\left\{\bar{n}^i \cdot \log \frac{K-1}{K}\right\}$$

which is the familiar exponential utility function, where $\bar{n}^i$ is the number of i's effective neighbors.

The function $f(.)$ is increasing in $\bar{n}^i$, bounded, and converges to 0 as the number of effective neighbors goes to infinity. It is easy to show that $f(.)$ exhibits (M, c)-strong decreasing where $M = \log\left(\frac{1-\sqrt{1-4c}}{2}\right) / \log\left(1 - \frac{1}{K}\right)$ if $c \leq 1/4$ and $M = 0$ otherwise.

[6] For simplicity, we ignore capacity constraints: each agent can solve as many problems as demanded.

2.2.2 Network Formation and the Value of Information

A central function of many social networks is the exchange of information. For example, Katz and Lazarsfeld (1955) show that social connections are a primary source of information for consumers. Similarly, Feldman and Spencer (1965) demonstrate that nearly two-thirds of new residents in a community relied on social contacts to choose a physician. Mintzberg (1973) finds that managers spend more than half of their time gathering information through personal sources.

To model these ideas, consider an economy where agents exchange information about a single unobserved underlying state variable $\theta \sim N(\mu, \tau^2)$. Agents wish to learn about θ because they have to choose an action $x^i \in \mathbb{R}$ that maximizes expected utility

$$\mathrm{E}\left[U^i(x^i, \theta)\right] = \mathrm{E}\left[-(x^i - \theta)^2\right].$$

Once θ is drawn, each player i receives a signal $z_i \sim N(\theta, \sigma^2)$. These signals are conditionally independent across agents.

Agents exchange information with neighbors in the network. The transmission of information is subject to frictions, perhaps because of perception errors, inaccurate transmissions or technological constraints. We model frictions by assuming that after communication, each player has access to noisy versions of the signals of players within a distance of d. Specifically, if player i is at a distance of $d^i_j \leq d$ from player j, then i receives the following signal from j:

$$\widetilde{z}^i_j = z_j + \sum_{k=1}^{d^i_j} \epsilon^i_{kj}$$

where z_j is player j's original signal, and the sum of the ϵ^i_k terms is the cumulative noise associated with transmission. The ϵ^i_{kj} are i.i.d. and normally distributed with mean zero and variance σ^2_ϵ. This reduced form can be obtained if there are d rounds of communication, where in each round players communicate with direct neighbors and transmit noisy versions of all the signals they observe.

With a slight abuse of notation, write $E[\theta|i]$ to denote the conditional expectation of θ given the values of all signals that player i observes, and $\mathrm{Var}(\theta|i)$ for the corresponding conditional variance. It is easy to see that the optimal action of player i is $x^i = E[\theta|\bar{n}^i]$, and that maximized utility equals $-\mathrm{Var}(\theta|i)$. Given our normality assumption, the conditional variance can be computed as

$$\mathrm{Var}(\theta|i) = \frac{1}{\pi + \rho\left(a_1 + \sum_{\delta=1}^{d} a_\delta n^i_\delta\right)} = \frac{1}{\pi + \rho a_1 + \rho \bar{n}^i}. \tag{3}$$

where $\pi = 1/\tau^2$ and $\rho = 1/\sigma^2$ are the precisions of the state and the signal, $a_\delta = \sigma^2 / \left(\sigma^2 + \delta\sigma^2_\epsilon\right)$, and $\bar{n}^i$ is defined as above. If $\sigma^2_\epsilon = 0$ then $a_\delta = 1$ for all $\delta \leq d$. If $\sigma^2_\epsilon > 0$, then a_δ is strictly decreasing in δ, implying that more distant neighbors add less in terms of signal precision, because more noise is accumulated during communication.

From an ex ante perspective, if the network is such that player i is paying for l^i links, her total expected payoff is given by $f(\bar{n}^i) - cl^i$ where the benefit function $f(\bar{n}^i)$ is given by (3). It is easy to verify that $f(.)$ exhibits (M, c)-strong decreasing returns with $M = \frac{1}{2\rho}\left[\frac{1}{c} - \frac{3}{2}\pi + \sqrt{\left(\frac{1}{c} - \frac{3}{2}\pi\right)^2 - 2\pi^2}\right]$ if $c \leq \left[(\sqrt{2} + \frac{3}{2})\pi\right]^{-1}$ and $M = 0$ otherwise.

3 Network Architecture and Welfare

3.1 Equilibrium Networks

Our goal is to determine the Nash equilibria of the network formation game described in Section 2. We begin by introducing some graph theoretic concepts. Two agents are *connected* to each other if there exists a path or sequence of links in the network between them. A *component* of the network is a maximal set of connected agents. The network is connected if all agents are connected to each other. The network in which no player maintains a link is called the *empty network*. An equilibrium is *non-empty* if at least one player maintains a link. The network architecture is a *periphery-sponsored star* if, as in Figure 1, one player, the center, maintains no links, and all other players maintain a single link to the center. The network is an *extended star* if the center maintains no links, all other players maintain a single link and are directly or indirectly connected to the center, and all players are at a distance of at most d from each other. For example, the network in Figure 2 is an extended star when $d \geq 5$.

Recall that $f(.)$ exhibits (M, c)-strong decreasing returns, and let $N_0 = (2M)^{2d+2}$. The following result characterizes equilibrium networks.

Theorem 1 *If $a_2 > a_3$ and $N > \max(N_0, 4)$ then the unique non-empty equilibrium architecture is a periphery-sponsored star. If $a_2 = a_3$ then for $N > \max(N_0, 2d + 1)$ any non-empty Nash equilibrium is an extended star.*

The proof proceeds using a series of lemmas. We state and prove the main lemmas in this section; more technical arguments are presented in the Appendix. First we establish that in equilibrium the network is tight: no two players are very far from each other. This result is a consequence of Assumption 1 about limits to long distance communication. Tightness implies that there exist players who have many direct neighbors. Next we establish that in a large network, each player maintains at most one link. This "one-link property" is a key result that follows because agents can access substantial benefits by forming a single link to some player with many direct neighbors. By Assumption 2, the benefit function exhibits strong decreasing returns and hence maintaining a second link is not optimal. Building on the one-link property, we use graph-theoretic arguments to show that in equilibrium the network is either a tree, or contains a unique directed circle. We

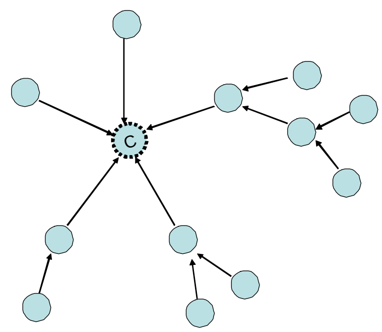

Figure 2: Extended star.

complete the proof using a revealed preference argument that is based on comparing the payoffs of terminal nodes, i.e., agents to whom no one links.

Lemma 1 *In any non-empty equilibrium there exists a player who is at most* $2d+1$ *far away from any other player.*

Proof. In a non-empty equilibrium, there are no isolated players. To see why, note that any isolated player k has the option to mimic some other player l who maintains at least one link. By concavity, the benefit of maintaining those links will be at least as high for k as it is for l. Moreover, by linking someone l is currently linked to, k gets the additional benefit of having indirect access to player l herself. So k strictly prefers to maintain some links.

Pick a player i who has the "highest paying $d-1$ wide neighborhood", that is, the player i with maximal $a_2 \cdot n_1^i + a_3 \cdot n_2^i + ... + a_d \cdot n_{d-1}^i$. Note that the indices are shifted relative to each other: i is the most attractive target from the perspective of an outside player. Suppose there exists an agent who is more than $2d+1$ far from i. This agent is not isolated, so either she maintains a link, or she is linked to by someone. In both cases, there exists a player j who is at least $2d+1$ far from i, and who maintains a link. Then j will find it beneficial to drop one of her links and maintain instead a new link to i. To see why, pick a link that j maintains to k. Clearly, k has a $d-1$ neighborhood that is at most as large as that of i. But the $d-1$-neighborhood of i does not overlap with the d-neighborhood of j and hence does not contain j herself. In contrast, the $d-1$ neighborhood of k does contain j, and hence k is less beneficial to link to than i. ■

Lemma 2 *For* $N > N_0$*, in any equilibrium all players maintain at most one link.*

Proof. If all players have at most u direct neighbors, then $N \leq u^{2d+2}$ because no agent is more than $2d+1$ away from the maximal player of Lemma 1 . Hence the largest 1-neighborhood in the economy will have be no smaller than $N^{1/(2d+2)}$. Then the effective number of people any player j who maintains a link has access to must be at least $a_2 \cdot N^{1/(2d+2)} = N^{1/(2d+2)}$, because j can choose to drop all links and link a player with the largest number of direct neighbors which already provides her with $a_2 \cdot N^{1/(2d+2)}$ effective neighbors. When j herself had the largest number of direct neighbors, she has access to even more effective neighbors.

Suppose there exists a player i who maintains more than one link. Let L be the set of players i is linking to, m the total effective number of people she has access to, and for each $j \in L$, let m_j be the effective number of people that i has access to only through j. Clearly, $\sum_{j \in L} m_j \leq m$. Because L has at least two elements, there is a j such that $m_j \leq m/2$. But then dropping the link to j is surely profitable if

$$\max_{N^{1/(2d+2)} \leq m < N} (f(m) - f(m/2)) < c$$

which holds by strong decreasing returns. ∎

The one link property restricts the network architecture the following way.

Lemma 3 *For $N > N_0$, if there exists a player who maintains no links, then the network architecture in a non-empty equilibrium is a directed tree, with the player maintaining no links at the center. If all players maintain a link, then the architecture contains a unique directed circle, and all agents in the circle are the endpoints of disjoint directed trees.*

Proof. Suppose that player i maintains no links. Then the players who are at distance 1 from i must all have a single link to i, and maintain no other links. Thus all their other neighbors have to be agents who maintain links to them. These are the players who are at distance 2 from i. Since these players again maintain no other links, their only other neighbors are the ones who maintain a single link to them. And so on. Clearly, in the process we eventually cover all players, because the network is connected. It follows that the network is a directed tree.

Next assume all players maintain a single link. Then the graph contains a directed circle that can be found by following any directed path, because the path can be continued at all nodes. Fix the directed circle: for each element of the circle we can repeat the above argument. Thus each element of the circle is the endpoint of a directed tree. These trees are disjoint, because all links maintained by the members of any of the trees stay within that particular tree. ∎

We proceed by examining the incentives of terminal nodes, i.e., players with no incoming links.

Lemma 4 *For $N > N_0$, if $a_2 > a_3$ then all terminal nodes maintain a single link to the same unique player.*

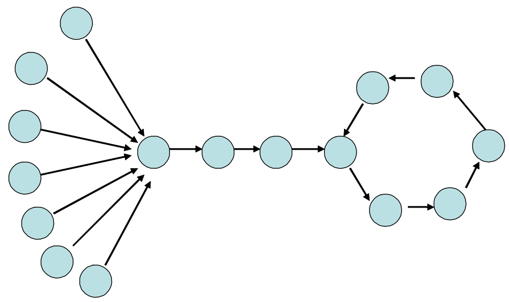

Figure 3: Star, directed path and circle network.

Proof. Suppose there exist two players i and j who are both terminal nodes. Because there are no isolated players, both i and j maintain a single link. Suppose that $\overline{n}^i \leq \overline{n}^j$ so that i has fewer effective neighbors and a lower payoff than j. Denote the distance between i and j by δ. Consider the deviation where i drops her single link and maintains a link to j's direct neighbor instead. By doing so, i will access $\overline{n}^j + a_2 - a_\delta$ effective neighbors. This is because i will access $n_2^j + 1$ agents at distance 2, where the $+1$ stands for player j. On the other hand, we need to subtract a_δ because $\overline{n}^j$ also included the benefit of j from indirectly knowing i if $\delta \leq d$. If $\delta > d$, we use the convention $a_\delta = 0$. Clearly, this deviation is profitable as long as $a_2 - a_\delta > 0$. When $a_2 > a_3$ the deviation is always profitable except when $\delta = 2$, but then the unique direct neighbor of i and j is the same by definition. ■

The lemma implies that in the case where $a_2 > a_3$, the network consists of a star, a single directed path and a directed circle. Figure 3 illustrates such a network architecture. To conclude, we need to show that the path and the circle are of zero length. We do this by showing (in the Appendix) that players in the path would find it beneficial to "leapfrog" those directly in front of them.

Lemma 5 *If $N > \max(N_0, 4)$ and $a_2 > a_3$, then the unique non-empty Nash equilibrium architecture is a periphery-sponsored star.*

If $a_2 = a_3$, then Lemma 4 no longer holds. In that case, a leapfrogging argument is used to prove that any two players in the network are at most d far away from each other. We then show that in a directed circle there always exists a player who benefits more from her incoming links than from the link she maintains. Given that this player is indirectly connected to everybody, she would find it optimal to maintain no links. This establishes that there are no directed circles, and leads the final lemma required to prove Theorem 1.

Lemma 6 *If* $a_2 = a_3$ *and* $N > \max(N_0, 2d+1)$*, then a non-empty equilibrium is an extended star. Moreover, if the maximum distance in the equilibrium is* d_1*, then it must to be that* $a_2 = ... = a_{d_1} = 1$*.*

How important are our two key assumptions for obtaining Theorem 1? First note that even if limits to communication and strong decreasing returns fail to hold, the periphery-sponsored star continues to be an equilibrium. The key content of these assumptions is that they guarantee uniqueness. Interestingly, uniqueness can be obtained even without limits to communication if we replace (M, c)-strong decreasing returns with the somewhat stronger $(1, c)$-strong decreasing returns assumption. To see why, note that limits to communication is used to establish Lemma 1, which is then used to show the players maintain at most one link. However, this one-link property can also be obtained directly under $(1, c)$-strong decreasing returns.[7] The logic is that any player maintaining two or more links has a link that provides at most half of her total benefit. But by assumption $f(b) - f(b/2) < c$ for any total benefit level $b \geq 2$, hence the cost of the "weakest link" outweighs the benefit. Given the one link property, it is easy to show that a non-empty equilibrium is connected, and as long as $a_3 < a_2$, the rest of the proof extends without modification. As a result, under $(1, c)$-decreasing returns, requiring a cutoff d limiting communication is not essential for our main theorem, although a weak form of discounting is needed.

3.2 Inequality, Efficiency and Welfare

The star architecture is asymmetric, since the center has a very different role from the players in the periphery. This asymmetry also manifests itself in terms of payoffs. The payoff advantage of the center relative to a player in the periphery is

$$[f((N-1)a_1) - f(a_1 + (N-2)a_2)] + c.$$

The first term in this expression measures the additional payoff the center derives from having *direct* access to the entire population. This term is large when a_1/a_2 is large, that is, when the communication technology is poor, and disappears with seamless communication. The second term derives from the fact that the center has access to all agents while maintaining no links. This term is small when link costs are small. It follows that improvements in network technology, whether

[7] Both examples in section 2 allow for this possibility.

they affect a_1/a_2 or c, result in a more egalitarian payoff distribution.[8]

Is the inequality in the equilibrium network associated with inefficiency? To answer, first note that the periphery-sponsored star is always Pareto efficient, because the center achieves the maximal payoff possible in this economy, and each player in the periphery achieves the maximum possible subject to the high payoff of the center. Hence there always exists a welfare function that is maximized by the Nash equilibrium network.

However, given that our network game is fully symmetric, it is natural to focus on a utilitarian welfare function with equal weights on all players: $\sum_{\bar{\imath}=1}^{N} \pi^i(g)$. Since the costs of links are separable, this definition of social welfare is not affected by how the costs of links are distributed. Using this welfare function Jackson and Wolinsky (1996) show that when $f(.)$ exhibits constant returns to scale, for intermediate costs of connections the star is the only efficient architecture. We show that with decreasing returns the results may be very different, for two reasons.

Example 1 *Suppose that N is even and let $f(n) = -1/(1+n)$, $c = 1$. Then the unique efficient network architecture has $N/2$ components, each with two players. The average payoff in the efficient network exceeds average payoff in the star by a term bounded away from zero for all N.*

The intuition can be seen by noting that in the network with $N/2$ components, the per capita link cost is only $1/2$, whereas in a connected network it would be $(N-1)/N$. When $f(.)$ is sufficiently concave, benefits do not grow at a fast enough pace to compensate for the higher per capita cost of links. This reasoning suggests that the efficiency of the star architecture requires a lower bound on the curvature of $f(.)$.

Example 2 *Suppose that N is even and $a_1 > a_2 = a_3$. Then for any concave benefit function, the network that consists of two identical and interlinked stars has a higher average payoff than the star.*

Here, the basic intuition is that averaging a concave function $f(.)$ yields a higher value when the arguments are more equal. The distribution of effective neighbors in the interlinked stars case is more equal, and hence the average payoff is higher. This example demonstrates that inequality in payoffs directly leads to an inefficient equilibrium independently of any restriction on the curvature of $f(.)$. The next theorem shows that inequality and curvature are an exhaustive list of the elements that prevent the equilibrium network from being efficient.

Theorem 2 *Assume that $f'(n) > 2ca_1^2/n^2$ for $n > a_1$ and that $N > f^{-1}(c + f(0))$, then*
(i) If $a_1 = a_2$ and $a_3 = 0$, the unique efficient network architecture is the star.

[8] This argument ignores the case when c is very large and the unique equilibrium is the empty network. In this case, a decrease in c can make the the periphery-sponsored star become an equilibrium, inducing an abrupt rise in inequality. Further decreases in the cost of communication are indeed associated with lower inequality.

(ii) For $N > \max\{N_0, 3a_1 - 2\}$ the average payoff advantage of the efficient network relative to the star is bounded above by $2c(1 - 1/a_1)/N$ and the number of components in an efficient network is bounded by a constant independently of N.

Part (i) shows that when $f(.)$ is not too concave and payoff inequality is limited by $a_1 = a_2$, the star is efficient. Since the benefit function used in Example 1 satisfies $f'(n) > 1/4n^2$ for $n > 1$, the result of the theorem is relatively sharp. To understand why the slope of $f(.)$ must scale with $1/n^2$ (or more), consider a star of size m. Since a star has $m - 1$ links, the average cost paid by an agent is $c\frac{m-1}{m}$, which has a slope equal to c/m^2. Hence, if benefits grow at some rate slower than c/m^2, it is more efficient to split agents in smaller stars than keep them together in a single component. Note that this argument does not rely on strong decreasing returns, hence (i) holds for any $f(.)$ satisfying $f'(n) > 2ca_1^2/n^2$.

Part (ii) shows that even in the presence of payoff inequality, the equilibrium is approximately efficient for N large. This is a consequence of strong decreasing returns, which make the benefit function relatively flat in the limit, bounding the effect of inequality in effective neighbors on inequality in payoffs.

3.3 Discussion

The results in this section show that in our network formation model, core-periphery structures emerge endogenously as a unique outcome, and that central agents achieve higher payoffs. Both of these predictions are consistent with the evidence about the organization of economic and social networks discussed in the introduction.

However, our basic setup has a number of limitations. In terms of assumptions, we have not allowed for heterogeneity in costs and benefits. We have also ignored the role of consent in link formation and the possibility of sharing link costs. In terms of results, our equilibrium rules out multiple centers, and our model is silent about the identity of the center. In the following sections, we consider three extensions of the basic model to address these issues.

4 Heterogenous Costs Yield Interlinked Stars

In this section we relax the assumption that all players in the network are homogenous. Assume that there are $\bar{T}$ types or groups of agents indexed by $t = 1, ..., \bar{T}$. Let the cost of maintaining a link between agents who are the same type be c, and denote the cost of maintaining a link between agents of different types by C, where $C > c$. The groups capture differences along socioeconomic characteristics like age, sex, race, geography, native language or intellectual background. The key underlying assumption is that it is easier to maintain a link between two people who are more alike along these dimensions.

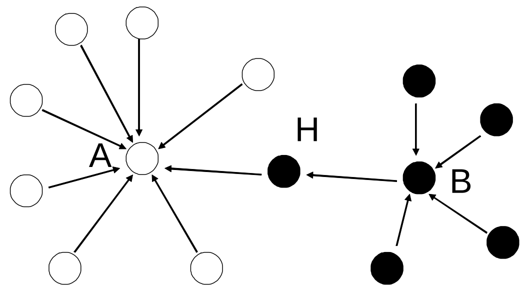

Figure 4: Interlinked stars.

For simplicity, throughout this section we assume that $f(.)$ exhibits $(1, c)$-strong decreasing returns, and that there are more than 4 agents of each type. We say that an equilibrium contains a *mixed component* if the equilibrium network has a component that has two different types of agents.

Theorem 3 *If $a_2 > a_3$, then any equilibrium that contains a mixed component has all players except for at most $5\bar{T} + 2d + 2$ organized in periphery-sponsored stars. These stars may be linked to each other, but there is at most one star for each type t and only one mixed component.*

In any equilibrium that contains no mixed component, for each t, either all type t agents organize a separate periphery-sponsored star, or all of them are isolated.

The proof is in the Appendix. The result shows that with heterogeneity, architectures more complex than the periphery-sponsored star may emerge in equilibrium. But except for a small number of players, a more complex equilibrium is still organized from building blocks that have a star architecture. Hence heterogeneity leads to a network of interconnected stars. One potential equilibrium with two types is shown in Figure 4.

To get some intuition about the proof, recall that $(1, c)$-strong decreasing returns directly imply the one-link property for every component. In contrast, if we were to use tightness to derive the one-link property as in Lemma 2, we would require lower bounds on the size of each component. The "universal" one-link property implies that the characterization result of Lemma 3 holds for *all* components in an equilibrium. As a result, each component is a combination of a circle and some directed trees.

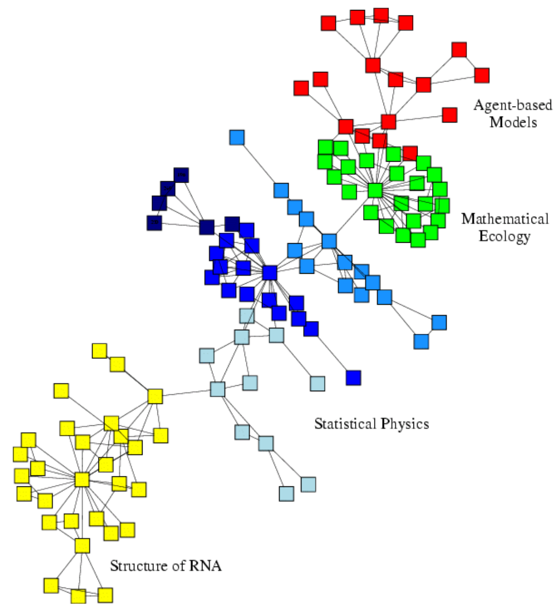

Figure 5: Scientific collaboration network (Newman, 2004)

Next we introduce the concept of a *bridge*: a player who maintains a link to some player of a different type. For example, player H in the equilibrium network of Figure 4 is a bridge. We generalize the revealed preference argument of Lemma 4 for bridges by showing that any two such agents who are independent in a well defined sense have to link to the same player. This implies in particular that there cannot be multiple mixed components, because each of them would contain an "independent" bridge. Moreover, in the single mixed component, all non-independent bridges must be organized in a single path. We conclude the proof by showing that all not too small homogenous components are stars by Theorem 1, and that the mixed component consists of subtrees connected to the "bridge path" which are homogenous stars because of a leapfrogging argument similar to Lemma 5.

While Theorem 3 does not rule out the possibility of a periphery-sponsored star, a richer set of networks can emerge. Equilibrium networks exhibit a modular structure, with agents of the same type organized around local centers with only a few links (if any) maintained between different communities. These predictions match important stylized features of some social networks.[9] As an example, Figure 5, taken from Newman (2004), illustrates a small network of coauthorship. The nodes of the network are scientists of a private research institute belonging to different disciplines. A link between two scientists indicates that they coauthored an article. It is clear from the picture

[9]Sociologists use the term homophily to refer to the extent to which individuals that communicate are similar (Lazarsfeld and Merton, 1964). Evidence on the role of homophily in networks can be found in Borjas (1992) and McPherson et al. (2001), among others.

that most collaborations stay within the same discipline, and that those disciplines which are more narrowly defined (e.g. Mathematical Ecology as opposed to Agent-Based Models) have a small number of prominent central agents who participate in many collaborations. Consistent with our model, the example suggests that there are relatively few bridges bridging disciplinary boundaries. Networks of scientific collaboration in the social sciences have similar structural characteristics: they are organized as interlinked "thematic" communities, each of which has central and peripheral researchers.[10]

4.1 Two Types

Under which circumstances should we expect a periphery-sponsored star, interlinked stars or fragmentation? What is the role of changes in the communication technology in shaping network structure? To answer these questions we now specialize to the case of two types ($\bar{T} = 2$). We call a network an *interlinked star* if each type is organized in a periphery-sponsored star, and the center of one of these stars either links the center of the other, or links an intermediate bridge who links the center of the other star. For example, the network in Figure 4 is an interlinked star. We say that the network is fragmented if each type is either organized in a (separate) star, or has all players isolated.

Corollary 1 *If $\bar{T} = 2$ and there are more than 4 players of each type, then any equilibrium is either a periphery-sponsored star, or an interlinked star, or is separated.*

The proof is in the Appendix. Building on Theorem 3, it is easy to show that in a mixed component, there are either one or two periphery-sponsored stars. The case with a single star can be handled using an easy revealed preference logic. With two stars, one star has to consist entirely of type 1, the other entirely of type 2 players, as in Figure 4. We conclude with a leapfrogging argument as in Lemma 5 to show that the path between the two centers is short.

The tractable two-type framework allows us to explore comparative statics. For simplicity, assume geometric discounting by distance, so that $a_i = a_1\beta^i$ for $i \leq d$, but note that the comparative statics results extend for arbitrary discounting with intuitive modifications. We also assume that $f(N_i) > C$, which rules out fragmented equilibria. Let ΔN stand for the absolute value of the difference between the size of the two groups. We will look at comparative statics holding fixed total group size N, and varying one of the parameters C, β and ΔN at a time.

Corollary 2 *Assume that $\bar{T} = 2$, $N_i > 4$ and $f(N_i) > C$ for $i = 1, 2$. Then there exist functions $\overline{C}(.)$, $\underline{\beta}(.)$, $n_0(.)$ and $n_1(.)$ such that*

(i) For $C < \overline{C}(c, \Delta N, \beta)$ the only non-empty equilibrium is a periphery-sponsored star.

(ii) For $\beta \geq \underline{\beta}(c, C, \Delta N)$ the only connected equilibrium is an interlinked star.

[10] See Moody (2004) and Goyal et al. (2005).

(iii) For N large enough, if $\Delta N \leq n_0(c, C, \beta)$ then the interlinked star is not an equilibrium. If $f(.)$ is unbounded, then for $\Delta N \geq n_1(c, C, \beta)$ separate stars are not an equilibrium.

The proof is in the Appendix. Corollary 2 can be used to study the effects of communication technology on network structure. By part (i), an improvement in network technology associated with a reduction in the communication cost C will increase the likelihood of a connected equilibrium. However, within the set of connected networks, different technological advances have opposing effects. A fall in C makes single stars a more likely outcome. On the other hand, (ii) shows that an increase in β favors interlinked stars. Intuitively, a fall in cost difference brings groups together; whereas improvements in communication allow them to continue maintaining a distance. This dichotomy of getting closer versus drifting apart as a response to different technological improvements has been pointed out by van Alstyne and Brynjolfsson (1996), Gasper and Glaeser (1998), and, more recently, by Rosenblat and Möbius (2004).

The result also shows how group size affects the shape of equilibrium networks. Assume that β is close to one, so that by (ii), a single star is not an equilibrium. In this case, (iii) shows that groups of relatively equal size are organized in separate and disconnected stars. On the other hand, if one group is substantially smaller, the equilibrium will be connected. These results are intuitive: large minority groups have little incentive to integrate, while small minorities integrate partially by forming an interlinked star. This is at the heart of Lazear's (1999) explanation for the differences in the assimilation of immigrants in 1900 vis a vis 1990.[11] Our model also suggests that in the case of partial integration, some agents may act as translators (player H in Figure 4). In the case of immigration, immigrants' children are likely to play this role.

To illustrate the proof of Corollary 2, consider the interlinked star of Figure 4. All type 1 (white) players link the white center *Amy*, and all black players except *Hannibal* link the black center *Bob*. Bob maintains a single link to Hannibal, who is a bridge and who links Amy. The fact that Bob maintains a link implies, by strong decreasing returns, that there are more type 1 (white) than type 2 (black) players. Consider the incentives of a type 2 agent in the periphery. If such an agent prefers to link Bob rather than Amy, it must be that the fixed cost of a link to Amy is high, because Amy provides access to more effective neighbors than Bob does. This explains (i). Result (ii) is shown by observing that when β is close to one, in a single periphery-sponsored star any type 2 agent would prefer to link another type 2 agent to save on link formation costs. Finally, to understand (iii), consider Bob's alternative strategy of dropping his link. If this is not beneficial, it must be that the gains from having access to type 1 players is substantial, which implies that ΔN is large. In a similar fashion, if the two stars are separate, an agent in the periphery of the smaller

[11] In 1900, the percentage of immigrants that learned English was 87%. In 1990, this number was down to 69%, despite the massification and improvements of communication technologies (including television). Lazear 's evidence shows that immigrants in 1990 were immersed in larger networks than immigrants in 1900, thereby facing a lower value of assimilation.

star would not migrate only if the additional benefits of the larger star are not too high, that is, if ΔN is low.

5 Centrality and Access to Resources

In this section we investigate the determinants of centrality in the long run. It is plausible to expect that agents who provide access to better or more critical resources should be more central in a network. Such heterogeneity can arise naturally in the examples of Section 2 if some individuals are better at solving problems or have access to better information.

This motivates us to extend the basic model of Section 3 by allowing one player to provide more valuable resources than the others. Denoting this player by 1, our formal assumption is that the payoff of any player i is

$$\pi^i(s^i, s^{-i}) = f\left(\sum_{\delta \leq d} a_\delta \cdot n^i_\delta + 1_{\{\overline{\delta} \leq d\}} \cdot a_{\overline{\delta}} \cdot \omega\right) - c \cdot l^i$$

where $\overline{\delta}$ is the distance between i and 1 and $1_{\{\overline{\delta} \leq d\}}$ equals one if this distance is not greater than d. The additional benefit of having access to player 1 is reflected in the parameter $\omega \geq 0$. The model of Section 3 corresponds to the special case with $\omega = 0$. Throughout this section, we continue to assume $(1, c)$-strong decreasing returns. To simplify the analysis, we also assume that $d = 2$.

Using the argument of Theorem 1, it is easy to show that as long as $a_1 > a_2$, for extremely large ω there is unique non-empty equilibrium, the periphery-sponsored star with player 1 as the center. Stars with other players in the center cannot be sustained in equilibrium, because agents would choose to connect directly to player 1. However, for ω positive but not too high, there continue to be N distinct non-empty Nash equilibria, depending on which player is the center. Thus the presence of an agent with better skills in itself need not reduce the multiplicity of static equilibria. Nevertheless, over the long term one might expect that the network with 1 as the center will be relatively more robust to evolutionary changes. To explore this idea formally, we study stochastically stable equilibria under a *perturbed best response dynamics* in the spirit of Kandori, Mailath and Rob (1993) and Young (1993).[12]

The main idea behind these dynamics is that most of the time, players are randomly called to adjust and choose a myopic best response to current play. However, with a small probability the agent called to best respond makes a stochastic mistake, and randomly chooses one of the available strategies. This stochastic mistake is called a "mutation." Our goal is to determine which networks prevail in the long run as the probability of mutations becomes vanishingly small.[13] Intuitively,

[12] This sort of evolutionary analysis has been used in other models of endogenous network formation by Goyal and Vega-Redondo (2005) and Hojman and Szeidl (2005).

[13] See Fudenberg and Levine (1998) for further discussion and justification of a best response dynamics.

such equilibria will be the most robust to small perturbations. For example, in 2×2 coordination games the only stochastically stable set corresponds to the risk dominant equilibrium, which is the unique best response when both opponent strategies are equally likely.

Formally, we introduce a family of Markov process Z_t^ϵ over the state space of pure strategy profiles $\times_{i=1}^{N} \{0,1\}^{N-1}$.

Definition 1 *For each $\epsilon \geq 0$ and initial state Z_0^ϵ, the evolutionary dynamics P^ϵ of the process Z_t^ϵ is defined as*

- *On each date $t \geq 0$ a single player is randomly selected to adjust. Selection probabilities are independent across players and time periods, and bounded away from zero and one for each player.*
- *If player i is selected to adjust, with probability $1-\epsilon$ she chooses a best response to the current state of the system. If the best response is not unique any best response is equally likely to be chosen. That is, $s_t^i \in \arg\max_{s^i \in \{o,1\}^{N-1}} \pi^i \left(s^i, s_t^{-i}\right)$ and $s_{t+1}^{-i} = s_t^{-i}$.*
- *With probability ϵ player i mutates by dropping at most one of her current links and forming at most one new link. Any of her current links is equally likely to be dropped (if at all) and, similarly, she is equally likely to form any new link (if at all).*

Some terminology related to stochastic stability will be useful (see Fudenberg and Levine, 1998). The dynamic with $\epsilon = 0$ is called the unperturbed or mutationless process. The recurrent classes of the unperturbed process P^0 are the limit sets of the dynamics. The basin of attraction of a limit set Ω, denoted by $D(\Omega)$, is the set of states from which the unperturbed dynamics reaches Ω with probability one. The unperturbed process is not ergodic; in particular all $N+1$ strict Nash equilibria of our network model are limit sets.

The dynamic P^ϵ with $\epsilon > 0$ is called the perturbed process. This is an irreducible Markov process which therefore has a unique invariant distribution μ_ϵ for each $\epsilon > 0$. Our goal is to determine the limit distribution $\mu^* \equiv \lim_{\epsilon \to 0} \mu_\epsilon$. It is easy to see that this limit distribution will be concentrated on the limit sets of the unperturbed dynamics. A limit set Ω is said to be stochastically stable if $\mu^*(\Omega) > 0$. If for some limit set Ω we have that $\mu^*(\Omega) = 1$, we say that Ω is uniquely selected in the long run. Our earlier intuition suggests that in the current model, the star with player 1 as the center is likely to be selected in the long term. The next result confirms this intuition.

Theorem 4 *Assume that $f(.)$ exhibits $(1,c)$-strong decreasing returns, $d = 2$ and N satisfies $f(\lceil (N-1)/3 \rceil) - f(1) > c$.*

(i) For N odd, the unique long term equilibrium for any $\omega > 0$ is the periphery-sponsored star with player 1 as the center.

(ii) For N even, the star with 1 as the center is uniquely selected for $\omega > 1/\left(a_1 - a_2\right)$.

Part (i) of the result says that in some cases, even an arbitrarily small additional benefit $\omega > 0$ can lead to centrality over the long run. Since central agents fare substantially better, this result suggests that small differences in endowments can be amplified and lead to large payoff inequality in the endogenous network formation process. Part (ii) has essentially the same content except that the additional benefit ω associated to player 1 has to exceed a threshold due to integer restrictions.

The proof is in the Appendix and relies on the radius-coradius theorem of Ellison (2000). By that result, we only need to show that the number of mutations required to transit from the limit set E^1 (the star with 1 as the center) to the basin of attraction of any other limit set exceeds the number of mutations required to come back. The argument is best illustrated by assuming that the other limit set is also a star E^2, with player 2 as the center.[14] Suppose that the system starts from E^1 and consider the sequence of m consecutive mutations where m players replace their link to player 1 with a link to player 2. After these mutations, the network has two interlinked, periphery-sponsored stars: the one with 1 as center has $N - m$ players, the other with 2 as center has $m + 1$ players. We can compute the radius as the smallest value m such that a player currently linking 1 would optimally migrate and link 2. Comparing the payoffs from staying and migrating yields $m = 1 + \lfloor (N - 3 + (a_1 - 1)\omega) / 2 \rfloor$, where the brackets represent lower integer value. The analogue exercise reverting the roles of players 1 and 2 yields $m' = 1 + \lceil (N - 3 - (a_1 - 1)\omega) / 2 \rceil$. This number is smaller because player 1 provides access to more valuable resources, implying that fewer mutations are required to escape the basin of attraction of E^2 than to escape the basin of E^1. Formally, $m \geq m'$ and the inequality is strict if either N is odd or $(a_1 - 1)\omega > 1$.[15]

6 Transfers

In this section we extend our basic network formation game to allow for transfers. A full characterization of equilibria in the setup with transfers is beyond the scope of this work. Our goal here is simply to explore conditions under which the periphery sponsored star continues to be the unique equilibrium architecture.

Formally, let the strategy of player i be a vector $t^i \in \mathbb{R}^{N-1}$, where t^i_j is the amount that i is offering to pay for a link between i and j. A link will be formed between i and j if and only if $t^i_j + t^j_i \geq c$. When this inequality holds strictly, the additional resources beyond the cost of the link

[14] Although not required for our proof, it can be shown that the only limit sets are the strict Nash equilibria of the model.

[15] The assumption that $f(\lceil (N-1)/3 \rceil) - f(1) > c$ holds implies that a player who has access to $\lceil (N-1)/3 \rceil$ others through her single link will not find it optimal to drop her link when she has a benefit of at least one player through an incoming link. This assumption rules out the possibility that the shortest path between E^1 and E^2 goes through the empty network. While relaxing the assumption would yield a similar selection result, the argument becomes significantly more involved, and requires the use of the radius-modified coradius theorem of Ellison.

c are wasted. Once links are created, a network emerges, and the payoff of a player i is given by

$$\pi^i(t^i, t^{-i}) = f(a_1 n_1^i + a_2 n_2^i + ... + a_d n_d^i) - \sum_{j:\ \exists \text{ link } (i,j)} t_j^i. \tag{4}$$

In the rest of the section, we refer to this network formation game simply as the "transfers game."

The cost sharing protocol described here is equivalent to the direct transfer network formation game proposed by Bloch and Jackson (2005). In contrast with our basic model with one-sided link formation, this protocol allows players to reject links. For example, player i can set t_j^i to be a very large negative number, preventing the link from being formed even if j is willing to pay the entire cost c or more. As we discuss below, the possibility of rejecting links plays an important role in the analysis.

Jackson and Wolinsky (1996) argued that in models with two-sided links and simultaneous announcements, Nash equilibrium fails to rule out some action profiles that appear implausible. Intuitively, in a Nash equilibrium, a new and mutually beneficial link between players i and j need not form, because it requires the consent of both parties. Jackson and Wolinsky proposed a refinement of Nash equilibrium, pairwise stability, which allows links that benefit both parties to be formed. Pairwise stability is often considered to be minimum requirement in network models with transfers.[16] In our model, pairwise stability is a relatively weak requirement. With strong decreasing returns, additional links are often costly; hence a new link between i and j may be of mutual interest only when either i or j can sever some existing links. To capture this idea, we introduce two new equilibrium concepts which are both refinements of pairwise stability. The first refinement is the following.

Definition 2 *A profile t is a pairwise Nash equilibrium with a unilateral status quo if*

1) it is a Nash equilibrium, and

2) for any action $\widetilde{t^i}$ of i and action $\widetilde{t^j}$ of j where $\widetilde{t_k^j} = t_k^j$ for all $k \neq i$, if

$$\pi^i(\widetilde{t^i}, \widetilde{t^j}, t^{-i-j}) > \pi^i(t^i, t^{-i}),$$

then $\pi^j(\widetilde{t^j}, \widetilde{t^i}, t^{-i-j}) < \pi^j(t^j, \widetilde{t^i}, t^{-i-j})$.

Beyond Nash equilibrium, this definition requires a profile to be robust to all deviations where player i makes arbitrary changes to t^i, proposes a link to j by modifying the transfer t_i^j, and j is better off accepting the link given the changes i made in her other links. A key aspect of this definition is that j, when deciding on accepting the proposed (i, j) link, assumes that i has already implemented the other changes in her strategy, and hence computes the potential gain of accepting relative to the profile $(t^j, \widetilde{t^i}, t^{-i-j})$. We say that the equilibrium has a "unilateral status quo"

[16] See Jackson (2005) and Calvo-Armengol and Ilkilic (2004) for a discussion.

precisely because if j rejects i's proposal, their outside option or "status quo" payoffs are computed in the network where player i has unilaterally severed some of her other links. Such outside options are sensible if it is very costly to re-establish a connection that has been severed, or when agents exhibit some degree of naivete and do not question the credibility of implausible threats. With unilateral status quo, agents never find it optimal to reject links in the sense of setting a negative transfer $t_i^j < 0$.

Definition 3 *A profile t is a pairwise Nash equilibrium with fixed status quo if*

1) it is a Nash equilibrium, and

2) for any action $\widetilde{t^i}$ of i and action $\widetilde{t^j}$ of j where $\widetilde{t}_k^j = t_k^j$ for all $k \neq i$, if

$$\pi^i(\widetilde{t^i}, \widetilde{t^j}, t^{-i-j}) > \pi^i(t^i, t^{-i}),$$

then $\pi^j(\widetilde{t^j}, \widetilde{t^i}, t^{-i-j}) < \pi^j(t^j, t^{-j})$.

In a pairwise equilibrium with fixed status quo we require the proposed (i, j) link to be beneficial to j relative to the old profile (t^j, t^{-j}). As discussed above, the fact that i can severe some links while offering a new link to j implies that she might make player j worse off in the new profile than she was in the original profile t. In a pairwise equilibrium with fixed status quo, j would then choose to reject the link proposed by i, which is why allowing agents to turn down link offers is a natural assumption in our framework. Note that in general, the pairwise equilibrium with unilateral status quo and the pairwise equilibrium with fixed status quo are not refinements of each other; they capture different notions of equilibrium. As we show in Theorem 5 below, in our application, pairwise equilibrium with unilateral status quo appears to be a stronger notion.

Given the exploratory nature of our analysis of the transfers game, in the rest of this section we focus on the case where $f(.)$ satisfies $(1, c/2)$-strong decreasing returns. The main advantage of this assumption is that it implies a variant of the one-link property irrespective of population size. Our analysis rests on an additional condition about the payoff structure.

Definition 4 *If the function*

$$f(a_1 + n) + f(a_1(n + 1)) - f(a_1 n)$$

is increasing in n for $n \geq 1$, then we say that $f(.)$ satisfies the monotone surplus condition (MS) with respect to a_1.

To understand the intuitive role of this condition, recall that in our original network formation game, a player choosing between connecting to one of two separate stars always finds it optimal to maintain a link to the center of the larger one. This property need not hold in the game with transfers. Intuitively, as link costs can be shared, the choice over the link to be formed is determined by the surplus the link creates for the two parties involved. For example, a link with the center of the smaller star may create a higher surplus if that center benefits substantially from an additional neighbor. Formally, the surplus from a new link between the isolated player and the center of a star with n neighbors equals

$$f(a_1 + a_2 n) - f(0) + f(a_1(n+1)) - f(a_1 n) - c,$$

where the first two terms are the gain of the previously isolated player, the next two terms are the gain of the center, and the last term is the cost of the new link. Noting that $a_2 = 1$, the surplus from connecting to a center with more neighbors will be higher if and only if this expression is monotone increasing in n, which is the condition stated in the definition above. This argument also shows how (MS) can be extended for more general benefit specifications than what we focus on here.

Example 3 *$f(n) = -A/(k+n)$ satisfies (MS) with respect to a_1 as long as $a_1 < k^2$. Moreover, $f(n)$ also satisfies $(1, c/2)$ strong decreasing returns when $2Ac < (2 + 1/k)(1 + 1/k)$.*

The example shows that while (MS) holds for a class of functions, it restricts the benefit structure further beyond strong decreasing returns. The next result shows that when (MS) holds, the core-periphery structure continues to be the unique equilibrium.[17]

Theorem 5 *(i) If $f(.)$ satisfies $(1, c/2)$ strong decreasing returns and (MS), $a_{\delta+1}/a_\delta$ is weakly decreasing for $1 \leq \delta \leq d-1$, and $N > 4d$, then any nonempty pairwise equilibrium with unilateral status quo has the architecture of a single star, in which players in the periphery pay at least $c/2$ for their link.*

(ii) If $f(.)$ satisfies $(1, c/2)$ strong decreasing returns and (MS), $d = 2$ and $N > 4 + 4a_1$, then any nonempty pairwise equilibrium with fixed status quo is either a single star, or two interlinked stars connected through their centers.

At the heart of the result is a key implication of (MS): if two players compete in attracting neighbors in order to become central, the player with more effective neighbors will be able to outbid his competitor. The proof formalizes this intuition using the idea that even in the presence of transfers, there is a natural sense in which links are directed. A link between players i and j is said to be directed from i to j if $t^i_j \geq c/2$, that is, if i pays at least half of the total link cost. In the case of equality, the direction of the link can be assigned arbitrarily. With this notion of directed links, using $(1, c/2)$-strong decreasing returns, it is easy to show that all players "maintain" at most one link. Thus each connected component of a pairwise equilibrium will have the tree and circle architecture described in Lemma 3. As in the proof of Lemma 6, directed circles are never a

[17] It is easy to construct an example where (MS) fails and multiple small stars are also an equilibrium.

feature of an equilibrium network. The key step in the proof is to establish a variant of Lemma 4 about terminal nodes. In part (i), (MS) allows us to show that the for any pair of terminal nodes, one has incentives to replicate the strategy of the other. The difficulty is that when a terminal node i proposes to connect to j, this change might reduce j's indirect neighborhood. In a pairwise equilibrium with unilateral status quo, i's threat to drop her old link is considered credible, and hence the indirect payoff effect does not enter j's calculation of the surplus. In part (ii), the same switching result obtains for any two terminal nodes at least $d+1=3$ far away from each other: In that case, there are no indirect payoff effects. This result implies that any pair of terminal nodes are at most 3 away. Combining the fact that the network is a directed tree with standard counting arguments allows us to complete the proof for both (i) and (ii).

7 Conclusion

This paper shows that limits in communication and strong decreasing returns lead to a core-periphery structure in endogenously organized networks. In a homogeneous population, a star architecture obtains uniquely, with the central agent having the highest payoff. This payoff inequality may render the equilibrium network inefficient. With heterogeneous groups, equilibrium networks feature subcommunities organized around local centers in a chain of interconnected stars. If one agent has more valuable qualities, noisy evolution selects the equilibrium with this agent is the center. Finally, when agents are allowed to bargain over link costs, under intuitive conditions the star continues to be the unique equilibrium.

An interesting question for future research is to model congestion by introducing capacity constraints on the number of connections agents can accept. Such an extension may lead to selectivity in accepting links and push the network towards a more hierarchical structure. A further, more ambitious research question is to study the interaction between network structure and the type of activity, such as coordination or cooperation that takes place between connected agents.

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Appendix

Proof of Lemma 5

By Lemma 3, the network is as depicted in Figure 3: there is a player i initiates a single directed path, which leads to a unique directed circle. Suppose that the network is not a directed circle itself. If i maintains no links, then we have a star. Otherwise, i links some player j. If the network outside of i and her direct neighbors contains at most 1 player, then by $N > 4$ and strong decreasing returns, either i or j would prefer to drop her single link. So j maintains a link to some k, and k also maintains a link. As long as j has no other direct neighbors, i would find it beneficial to drop her link to j and maintain a link to k instead, in effect "leapfrogging" j. By doing so, the size of i's 1-neighborhood is unchanged (k replaces j), while the size of her 2-neighborhood increases (previously it contained k, now it contains j and k's other direct neighbor), moreover, by leapfrogging, she would get closer to all other players in the path and the circle. This is a contradiction.

The only remaining possibility is when j has other direct neighbors besides i and k. Because this part of the network consists of a single path leading to a directed circle, it has to be that j has a single other incoming link from some player l, and this link closes the circle. In other words, i is maintaining a link to the directed circle. Consider the deviation by l of dropping her link to j, and instead linking i. This deviation essentially increases the size of the circle. After the deviation, all terminal nodes linking i, who used to be 3-neighbors of l will become 2-neighbors. It follows that this deviation is profitable as long as $a_2 > a_3$. The contradiction proves that the network consists only of i and the terminal nodes directly linking her. Finally, in case the network is a directed circle, each player would find it beneficial to leapfrog the agent she maintains a direct link to, because $N > 4$. Note that for $N = 3$ and $N = 4$, a directed circle can be an equilibrium network.

Proof of Lemma 6

Because $a_2 = a_3$, we have $d > 2$. It is easy to see that when $N > 2d + 1$, the circle network is not an equilibrium. Pick two players, i and j, such that the distance between them in the graph is maximal. The network is not a circle, therefore one of the two players, player j, is a terminal player who maintains a single link to k. Consider all other players who link to k. These players all have to be terminal, because they are exactly as far away from i as j is, which is the maximum possible distance. If k maintains no links then the network is a star, and in particular any two players are at most 2 far away from each other. Otherwise, k maintains a single link to some player l. Now consider the deviation in which j replaces her link to k with a link to l. Following the deviation, the size of j's 1-neighborhood is unchanged (l replaces k), while the joint size of her 2-neighborhood and 3-neighborhood has weakly increased. Previously this neighborhood consisted of all direct neighbors of k and l; now it still contains those players, but potentially more. There may be fewer 2-neighbors and more 3-neighbors now, but because $a_2 = a_3$, this does not matter. Moreover, j gets closer to all the rest of the players she used to access via l: Previously j accessed that part of the graph only via k and l; now she accesses it directly via l. The only case when this extra step is not gaining more neighbors to j is when (a) she already had access to the whole network previously, and (b) the a_δ weights corresponding to all the people she has indirect access to are equal. Because of (a), her distance from player i was not more than d; hence the maximal distance d_1 in the network satisfies $d_1 \leq d$.

Next we show that the equilibrium network never contains a circle. This is because there will always be a player i along the circle who gets more effective neighbors from her incoming link along the circle than from the link she maintains. If such a player i exists, by the above argument, i has at least $N-1$ effective neighbors, and hence strong decreasing returns ($N > N_0$) implies that she will optimally drop her link. To see why such a player i exists, for each player j in the circle define ϕ^j – player j's *forward advantage* – to be the difference between the effective neighbors from the link she maintains minus the effective neighbors from her incoming links. More precisely, let δ be the distance between j and some other player k in the network. If distance is attained by path(s) that contains the link maintained by j, assign a weight $w^{jk} = a_\delta$ to k; otherwise, set $w^{jk} = -a_\delta$. The forward advantage of j is $\phi^j = \sum_{k \neq j} w^{jk}$. We claim that $\sum_{j \text{ in the circle}} \phi^j \leq 0$. Note that if the latter holds there must be at least one player i whose forward advantage is non-positive and strong decreasing returns implies that this player should drop her link.

To prove the claim note that ϕ^j can be decomposed as the sum of two parts: $\phi^{j,circle} = \sum_{k \text{ in the circle}} w^{jk}$ and $\phi^{j,out} = \sum_{k \text{ outside the circle}} w^{jk}$. Clearly, $\phi^{j,circle} \leq 0$; so we just need to show that $\sum_{j \text{ in the circle}} \phi^{j,out} \leq 0$. To see why the latter holds, consider a player k outside the circle and let l be the player in the circle who is the root of the sub-tree that contains k. Note that $w^{l,k} \leq 0$. Let j_1 and j_2 be two players in the circle each at distance $\delta \leq \overline{\delta}$ from l (also in the circle), where $\overline{\delta}$ is the maximum distance between two players in the circle. That is, j_1 and j_2 are equidistant and "opposite sides" from l. By construction $w^{j_1 k} + w^{j_2 k} = 0$. It follows that $\sum_{j \text{ in the circle}} w^{jk} \leq 0$, and summing over all players k outside the circle gives $\sum_{j \text{ in the circle}} \phi^{j,out} \leq 0$.

Proof of Examples in Section 3.2

Example 1. Writing $v(m)$ for the maximal average payoff of a connected component with m players,

$$v(m) \leq f(a_1(m-1)) - \frac{m-1}{m}c \tag{5}$$

holds because such a component must have at least $m-1$ links. For $m = 2$, equality is achieved. It is easy to verify that the chosen parameters imply $v(m) < v(2)$ for $m \neq 2$, which proves that the unique efficient network involves $N/2$ isolated pairs. Because $\sup_{m>2} v(m) < v(2)$ also holds, the payoff advantage is bounded away from zero for any network size.

Example 2. Note that average link costs are equal in these two networks. Because $a_2 = a_3$, players in the periphery achieve the same payoff in either network. Thus to evaluate efficiency, we need to calculate the average payoff of the two central agents in the interlinked star and compare that with the average payoff of the central agent and a player in the periphery in the single star. It is easy to see that these pairs have access to the same number of effective neighbors in the two networks, but the distribution of effective neighbors is unequal in the single star and equal in the interlinked star. Because the benefit function is concave, average payoff will be higher in the interlinked star.

Proof of Theorem 2

Any component of size m in an efficient network must have a per capita payoff that equals $v(m)$, the highest per capita payoff across all connected networks of size m. As a result, the social welfare of any network with N agents is at most $N \cdot \max_{m \in \{1,\ldots,N\}} v(m)$

Write $\underline{v}(m) = \frac{m-1}{m}[f(m-2+a_1)-c] + \frac{1}{m} f(a_1(m-1))$ for the average payoff of m agents organized in a star architecture. The assumption that $N > f^{-1}(c+f(0))$ implies that $\underline{v}(N) > v(1)$ and hence rules out the case in which the empty network is efficient.

(i) Consider the set of connected networks of size m. When $a_1 = a_2$ and $a_\delta = 0$, $\delta \geq 3$, the star is an efficient architecture in this set. This is because any component must have at least $m-1$ links, and a star attains the maximal benefits as all agents are connected to each other. It follows that $v(m) = \underline{v}(m)$.

Using $f(a_1(m-1)) - f(m-2+a_1) < (a_1-1)(m-2)f'(m-2+a_1)$ which follows by strict concavity, it is easy to show that $f'(n) \geq 2ca_1^2/n^2$ implies $\underline{v}'(m) > 0$ for all $m \geq 2$. But then $v(m)$ is strictly increasing, and the upper bound $N \cdot \max_{m \in \{1,\ldots,N\}} v(m) = N \cdot v(N)$, which is attained by the periphery sponsored star with N agents.

(ii) For each $m \geq 2$, let $\underline{z}(m) = \frac{m-1}{m}$, $\overline{z}(m) = \frac{m-1}{2}$ and define

$$\overline{v}(m) = \max_{z \in [\underline{z}(m), \overline{z}(m)]} f(m-1+2z(a_1-1)) - cz. \tag{6}$$

The proof is structured in four steps.

Step 1: $\overline{v}(m) > v(m)$. Let x be the number of links in an efficient component of size $m \geq 2$. Let x^i respectively z^i denote the number of direct and indirect neighbors of player i in this component. Player i's benefits are bounded above by $f(z^i + a_1 x^i) \leq f(m-1+(a_1-1)x^i)$, which implies that total welfare in this component

$$mv(m) \leq \sum_i f(m-1+(a_1-1)x^i) - cx.$$

By strict the concavity of $f(.)$, using $\sum_i x_i = 2x$,

$$v(m) < f\left(m-1+(a_1-1)\frac{\sum_i x^i}{m}\right) - c\frac{x}{m} = f\left(m-1+2(a_1-1)\frac{x}{m}\right) - c\frac{x}{m} \leq \overline{v}(m),$$

since $m-1 \leq x \leq m(m-1)/2$ by definition.

Step 2: $\overline{v}'(m) > \left(\frac{2ca_1^2}{(a_1-1/2)^2} - c\right)/m^2$ and in particular $\overline{v}(m)$ is strictly increasing. To understand the shape of $\overline{v}(m)$, note that the maximization in (6) is a concave problem and the domain over which the maximum is taken shifts out as m increases. This implies that $\overline{v}(m)$ has three regions: (R1) for m small, $\overline{v}(m)$ is determined as a corner solution where the maximum in (6) is obtained for $z = \overline{z}(m)$; (R2) for intermediate m, the maximum is obtained as an interior solution; (R3) for m large the maximum is attained when $z = \underline{z}(m)$ again a corner solution. For each of these cases, $\overline{v}'(m) > \left(\frac{2ca_1^2}{(a_1-1/2)^2} - c\right)/m^2$ can be verified directly, using $f'(n) \geq 2ca_1^2/n^2$, and the envelope theorem in the interior case.

Step 3: When $m > \max\{2M + 1 - a_1, 3a_1 - 2\}$, $\overline{v}(m)$ is in region (R3), in particular $\overline{v}(m) = f(m-1+2\underline{z}(m)\cdot(a_1-1)) - c\underline{z}(m)$ since the expression in (6) attains the maximum where $z = \underline{z}(m)$. To see why, note that using $\underline{z}(m) \geq 1/2$,

$$f'(m-1+2\underline{z}(m)(a_1-1)) \leq f'(m-2+a_1) < \frac{f(m-2+a_1)-f((m-2+a_1)/2)}{(m-2+a_1)/2} < \frac{2c}{m-2+a_1} \leq c$$

when $m - 2 + a_1 > M$ and $m \geq 3a_1 - 2$ because of strong decreasing returns. But this chain of inequalities implies that (6) is maximized for $z = \underline{z}(m)$.

Step 4: It follows from the argument so far that the per capita payoff in the efficient network is at most $\overline{v}(N)$. Then we can bound the inefficiency of the star by

$$\begin{aligned}
\overline{v}(N) - \underline{v}(N) &= f(N-1+2\underline{z}(N)(a_1-1)) - f(N-2+a_1) \\
&\leq f(N-3+2a_1) - f(N-2+a_1) \leq (a_1-1)f'(N-2+a_1) \\
&< \frac{(a_1-1)\cdot 2c}{N-2+a_1} < \frac{2c(a_1-1)/a_1}{N}
\end{aligned}$$

as desired, where we used the assumption of strong decreasing returns.

Suppose that an efficient network has Q components and $m_1, ..., m_Q$ players in each of these components. Let $W = \sum_{q=1}^{Q} m_q v(m_q)$ denote the associated social welfare. We have

$$\underline{v}(N) - v(m_q) \geq \overline{v}(N) - v(m_q) - \frac{u}{N} \geq \overline{v}(N) - \overline{v}(m_q) - \frac{2c(a_1-1)/a_1}{N} \geq \left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right]\frac{(N-m_q)}{Nm_q} - \frac{2c(a_1-1)/a_1}{N}$$

where the last inequality follows from $\overline{v}(N) - \overline{v}(m_q) = \int_{m_q}^{N} \overline{v}'(n)dn \geq \left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right]\frac{(N-m_q)}{Nm_q}$. Multiplying by m_q and summing over q yields

$$N\underline{v}(N) - W \geq \left[\frac{\kappa}{(a_1-1/2)^2} - c\right](Q-1) - 2c(a_1-1)/a_1.$$

Since $N\underline{v}(N) \leq W$ by definition, it follows that $\left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right](Q-1) - 2c(a_1-1)/a_1 \leq 0$ and hence $Q \leq 1 + 2c(1-1/a_1)\left[\frac{2ca_1^2}{(a_1-1/2)^2} - c\right]^{-1}$ a constant.

Proof of Theorem 3

Consider an equilibrium that has a connected component Σ containing $N_\Sigma > 4$ players who are of $T > 1$ different types. For simplicity, we denote the types appearing in Σ by $t = 1, 2, ..., T$.

As discussed in the text, by $(1, c)$-strong decreasing returns the one link property holds in Σ. By Lemma 3, the single link property implies that Σ is either a directed tree, or a directed circle and a number of directed trees ending in that circle. By the argument in Lemma 6, there is no directed circle in Σ. For a player i in Σ, define the subtree of i to be the directed tree that has i as the endpoint. The subtree contains i, set of players linking i, the set of players linking this latter set, and so on. We say that i and j have independent subtrees if i is not contained in the subtree of j, and j is not contained in the subtree of i.

Lemma 7 *If $a_2 > a_3$, a pair of players i and j have independent subtrees in Σ, and either*

a) both of them are bridges, or

b) they have the same type and both of them are terminal nodes

then i and j maintain a single link to the same player.

Note that in part b), the restriction that both i and j are terminal nodes already implies that they have independent subtrees.

Proof. a) Let n_i be the number of players i has access to through her single link, not counting the agents that are in the subtree of j, and define n_j symmetrically. Let b_i and b_j be the effective number of agents i and j have access to in their own subtrees. Suppose that $n_j \geq n_i$, and consider the deviation by i of dropping her single link and instead replicating j's link strategy. Since i is a bridge, the new link costs at most as much as the old link did. By deviating, i will access n_j instead of n_i people from the set of agents outside of both i's and j's subtrees. Further, i will be closer to the people in j's subtree, including j herself, and i will continue to have access to all the b_i people in her own subtree. As long as $a_2 > a_3$, i strictly gains from being closer to the people in j's subtree, unless she was already linking j's direct neighbor. In equilibrium a deviation is never profitable, so i and j link the same player.

b) Let δ be the distance between i and j, and let c_i respectively c_j be the costs i and j are paying for their current links. Consider the deviation where i replicates j's link decision. Since this is not profitable,

$$f(n_i + a_\delta) - c_i \geq f(n_j + 1) - c_j$$

where we used that both i and j are terminal nodes, hence they earn no payoff from people linking them, and that they are the same type, so their costs of linking any given agent are the same. The symmetric deviation for j implies

$$f(n_j + a_\delta) - c_j \geq f(n_i + 1) - c_i$$

and adding the two inequalities yields

$$f(n_i + a_\delta) + f(n_j + a_\delta) \geq f(n_j + 1) + f(n_j + 1)$$

a contradiction unless $\delta = 2$ in which case i and j are currently linking the same player. ■

The subtree of a player i is homogenous if all players in the subtree are the same type.

Lemma 8 *If $a_2 > a_3$ and the subtree of i is homogenous, then either all players in the subtree link i (in which case the subtree is a star) or there is a player j linking i, and all other players in the subtree link j. In the latter case, i has to be a bridge.*

Proof. By Lemma 7 part b), all terminal nodes in i's subtree link the same player j. Thus the structure of the subtree is the following: a number of players link j, who initiates a directed path to i, and players in this path are not linked by any other players. We have to show that the path has length zero or one. Suppose that j links a player $k \neq i$, and k links l (who may be i). If l maintains no links (i.e., $l = i$ and this is the whole network) then by strong decreasing returns and $N_\Sigma > 3$, it is easy to show that k would drop her single link. Thus we know that l also maintains a link. In this case, by $a_2 > a_3$, j would find it beneficial to replace her link to k with a link to l, in effect "leapfrogging" k. This cannot be an equilibrium, hence the path consists of at most one player j. Finally, the only case when j would not leapfrog and connect to i's other direct neighbor instead of i is when this involves an increase in costs, in other words, when i is a bridge. ■

These results restrict the structure of Σ the following way.

1) There is a "bridge path": Lemma 7 part a) implies that all bridges in Σ are in the same directed path, except that there may be multiple bridges linking the same player at the origin of the path. To see why, note that for any two bridges i and j who are not linking the same player, one of them is contained in the subtree of the other, which defines a unique directed path between them. Any other bridge has to be either "above", or "below" or "in" that path, otherwise she would have independent subtrees with i or j. As long as there are bridges above or below the path, we can extend the same reasoning to generate a longer path encompassing all bridges involved. Moreover, if there are two bridges k and l linking the same player, then all bridges h not linking this same player have to be "above" them, because the subtrees of k and l are disjoint, hence this is the only way to avoid h having an independent subtree with either k or l.

2) There is at most one maximal homogenous subtree for each type, because any two terminal nodes of type t link the same player. If the homogenous subtree of i is not a periphery-sponsored star, then i is a bridge, hence she is either in the directed circle or in the bridge path.

3) The subtrees of the bridges who link the same agent j at the end of the bridge path are all homogenous. If any of these bridges is a terminal node (of type t), then all other terminal

nodes of type t are also linking j by Lemma 7 b). It follows that all players of type t not in the bridge path are linking j, because they have to be organized in a homogenous tree, and they cannot have terminal nodes linking a player different from j. As a result, these players are organized in a periphery-sponsored star with center j.

This discussion implies that with a few exceptions, all players in Σ are organized in periphery-sponsored stars. The exceptions are: 1) players in the bridge path; and 2) at most one player for each type, who is the bridge of the corresponding homogenous subtree.

To bound the number of players in the bridge path, let i be the player with the "highest paying $d-1$ wide neighborhood" in the component, that is, the player i_t with maximal $a_2 \cdot n_1^i + a_3 \cdot n_2^i + ... + a_d \cdot n_{d-1}^i$. By the argument of Lemma 1, all bridges in the network are at most $2d+1$ away from this player. Hence, the length of the bridge path is at most $2d+2$. It follows that the number of exceptions in groups 1) and 2) are at most $2d+2+T$.

We now determine the structure of all components in the network. If there is a player i of type t who maintains a link, then no type t player j is isolated, because she could earn positive payoff by replicating the link strategy of i. Likewise, if there exists a bridge in the network, then no player is isolated. Consider now all connected components of the network, Σ_1, Σ_2,...,Σ_r. By the argument of Lemma 7 part a), there cannot be bridges i and j in different components, because they need to link the same player. This shows that all components except at most one are homogenous.

Other then isolated players, there cannot be more than one homogenous component of type t by Lemma 1. Moreover, if there is a homogenous component of type t that has more than one element, then no type t agent is isolated. It follows that other than isolated players, there are at most $\bar{T}+1$ components, and all of them except one are homogenous.

A homogenous component of size larger than 4 is a periphery-sponsored star by Theorem 1. This shows that except for $5\bar{T}+2d+2$ agents, all non-isolated players in the network are organized in periphery-sponsored stars. There is at most one star for each type; and stars may be linked to each other or separate, but there is at most one component that contains agents of different types. If such a component exists, than no player is isolated.

Proof of Corollary 1

The result is immediate for an equilibrium that contains no mixed component. If there is a mixed component, then by the argument in the proof of Theorem 3, it contains a bridge path which includes two players, i and j, who are the only ones with incoming links from outside of the path. If $i=j$ then all terminal nodes link the same player, and an argument similar to Lemma 5 shows that the component is a single star. Otherwise, there are no terminal node bridges, which by Lemma 8 implies that all type 1 terminal nodes directly link a player i_1 who links i, and all type 2 terminal nodes directly link a player j_1 who links j (it is possible that $i_1 = i$ or $j_1 = j$). Moreover, the bridge path has to be a directed path between i and j: by the usual leapfrogging and strong decreasing returns argument, it cannot extend longer in either direction. Suppose that the flow of the path is from i to j. If player j_1 exists, then it has to be that j and j_1 are of the same type 2. But in this case, whoever links player j in the bridge path would find it optimal do link to j_1 instead (because by assumption j_1 has at least one incoming terminal node link). This shows that $j_1 = j$. If i_1 exists, then relabel her to be i: the component thus consists of a directed path between i and j, with all type 1 terminal nodes linking i and all type 2 terminal nodes linking j. By leapfrogging, the length of the directed path can be at most three: if i is not linking j directly, it has to be that i links some player k (who is of type 1) and k links j. Thus the mixed component has the structure outlined in the theorem. In the presence of a mixed component there cannot be a homogenous component of any type t, because there would be terminal nodes of type t in both the mixed and the homogenous component, contradicting Lemma 7.

Proof of Corollary 2

(i) Consider an interlinked star, and denote the centers of the two groups by k_1 and k_2. By the argument in the text, the bridge is a member of the smaller group. Assume this is the group of type 2 agents. By revealed preference, a player in the periphery of the second group finds it suboptimal to replace her current link with a new link to player k_2. Assuming that the interlinked star is such that k_2 links k_1 directly, the formal condition for this is

$$f\left(1+\beta\left[N_2-2\right]+\beta^2+\beta^3\left[N_1-1\right]\right) > f\left(1+\beta\left[N_1-1\right]+\beta^2+\beta^3\left[N_1-2\right]\right)-\Delta C,$$

which is violated for $\Delta C = 0$. By continuity, the inequality will be violated as long as $C < \overline{C}(c, \Delta N, \beta)$ for some threshold level $\overline{C}(c, \Delta N, \beta)$. A similar bound can be derived for the other potential interlinked star architecture involving an intermediate bridge (as in Figure 4). Separate stars can be ruled out the same way, and fragmented but non-empty equilibria do not exist by $f(N_2) > C$.

(ii) Consider the periphery-sponsored star with a type 1 center, and focus on a type 2 agent in the periphery. This agent does not replace her current link with a link to another type 2 as long as

$$f\left(1+\beta\left[N-2\right]\right)-C > f\left(1+\beta+\beta^2\left[N-3\right]\right)-c$$

holds. This inequality will be violated for $\beta = 1$, proving the existence of the desired threshold $\underline{\beta}(c, C, \Delta N)$.

(iii) Consider an interlinked star with no intermediate bridge. The center k_2 does not drop her single link only if $f(N_2+\beta\left[N_1-1\right])-C > f(N_2-1)$. If $N_1 = N_2$ this is equivalent to

$$f(N/2-1+1+\beta\left[N/2-1\right])-C > f(N/2-1),$$

a condition that will be violated for N large by $(1, c)$-strong decreasing returns. As a result, for N large we have a threshold for ΔN as desired. A similar argument applies for the interlinked star with an intermediate bridge.

To prove the second part of (iii), take separate stars with centers k_1 and k_2. A player in the periphery of the second star does not deviate only if

$$f(1+\beta\,[N_1-1]) - C < f(1+\beta\,[N_2-2]).$$

Given that $f(.)$ is unbounded, this condition cannot hold for N large enough and ΔN large.

Proof of Theorem 4

We first introduce a few standard definitions from the literature on stochastic stability. The cost $C(Z, Z')$ of a transition from Z to Z' equals $\lim_{\varepsilon\to 0}[\log P(Z'|Z)/\log \varepsilon]$ where $P(Z'|Z)$ is the probability of a transition from Z to Z' in one period. The cost of a path $Z = (Z_1, Z_2, ..., Z_k)$ is the sum of the costs of individual transitions between consecutive states Z_t and Z_{t+1}. In our case, the cost is simply the number of mutations taking place along the path. The radius $R(\Omega)$ of Ω is the minimum cost necessary to leave $D(\Omega)$ if play starts in Ω. The coradius $CR(\Omega)$ is the maximum across all states of the minimum cost required to reach $D(\Omega)$. Ellison (2000) shows that if $R(\Omega) > CR(\Omega)$ then Ω is the unique stochastically stable set. Our goal is to prove this inequality for $\Omega = E^1$.

We first establish that under $(1, c)$-strong decreasing returns, any limit set other than the empty network consists only of states where all players maintain at most one link. To see why, pick a state Z in a limit set and consider a path along the unperturbed dynamics in which each player best responds once. Any such path has positive probability. By $(1, c)$-strong decreasing returns, each player who best responds along this path will choose to maintain at most one link. At the end of this path, all players maintain at most one link; moreover, the argument also shows that the single link property is preserved over best responses. Because from any state we get to a state with the single link property with positive probability, it follows that limit sets consist only of such states.

Lemma 9 *The radius of the star with player* 1 *as the center satisfies*

$$R(E^1) \geq 1 + \min\left\{\left\lfloor \frac{N-3}{2} + \omega\frac{a_1-1}{2}\right\rfloor, \left\lfloor\frac{N-1}{2}\right\rfloor\right\}.$$

Proof. Let

$$z = \min\left\{\left\lfloor \frac{N-3}{2} + \omega\frac{a_1-1}{2}\right\rfloor, \left\lfloor\frac{N-1}{2}\right\rfloor\right\}$$

and consider a path starting from the star with 1 as the center.We claim that along any path with $y \leq z$ mutations, (a) all players who best respond will either choose a single link to 1, or decide to maintain no links, and (b) player 1 has at least $(N-1)/3$ direct neighbors. We prove this result by induction on y. For $y = 0$ the claim is immediate. Assume the result is true for up to $y-1$, and consider a path in which the last step is the y-th mutation. We make three observations about the structure of the network at this stage.

1) By the inductive assumption, any link not involving player 1 in this path was formed as a result of a mutation. Consequently, at most y new links have been formed.

2) The deletion of a link between some player k and player 1 can be the result of either a mutation or a best response. If k chooses to delete her link to player 1 as a best response, she must have at least two (direct or indirect) incoming links. This is because by (b) player 1 still has $(N-1)/3$ or more direct neighbors, and by $f(\lceil (N-1)/3 \rceil) - f(1) > c$ dropping such a link is not worthwhile when one has a single incoming link. Both of k's incoming links have to be the result of mutations.

3) Consider a player $j \neq 1$. Any direct neighbor of j except potentially 1 has to come from a mutation by 1). Moreover, by 2), a mutation that resulted in the deletion of some link between k and 1 does not increase the direct neighborhood of j, since such a mutation created a link between the mutating player and player k.

These observations imply that best-response link-deletions place a bound on the size of the neighborhood of any player j. Specifically, if l links to player 1 were deleted as a best response, any player $j \neq 1$ can have at most $y - 2l + 1$ direct neighbors. On the other hand, player 1 continues to have at least $N - 1 - y - l$ direct neighbors. Note that 2) implies $l \leq y/3$, which immediately gives that player 1 continues to have at least $(N-1)/3$ neighbors after the y-th mutation.

Now consider the incentives of a player i who comes to best respond after the y-th mutation. We know that any player forms at most one link as a best response, so to prove a), we only need to rule out the possibility that i links some player $j \neq 1$. Suppose that i is a player who maintains a direct link to 1. By the above bound on the direct neighbors of 1, i currently has access to at least $a_1(1+\omega) + N - 2 - y - l$ effective neighbors. Consider the alternative of linking player j. By doing so, i will gain access to at most $a_1 + y - 2l + 1 + \omega$ effective neighbors (the ω term is there because she might indirectly access 1). Maintaining a link to 1 is better if

$$a_1(1+\omega) + N - 2 - y - l > a_1 + y - 2l + 1 + \omega$$

equivalently

$$y < \frac{N-3}{2} + \omega \frac{a_1 - 1}{2} + \frac{l}{2}$$

which holds by definition as long as $y \leq z$, because $l \geq 0$. Hence i will not choose to link a player different from 1. If i does not maintain a link to 1, then the relative payoff benefit from linking to

1 as opposed to j will be even higher. In that case, the gain from linking 1 will increase because we no longer count i as one of the neighbors of 1.

It follows that immediately after the y-th mutation, any player who best responds will maintain a single link with 1 or maintain no links. But this implies that observations 1), 2) and 3) remain valid one step after the y-th mutation as well. As a result, the second step along the unperturbed dynamics will also involve a link with 1 or no link, meaning that properties 1), 2) and 3) remain valid two steps after the y-th mutation too. This argument can be repeated arbitrary many times, giving the proof of both a) and b) in the inductive step for any path that involves at most y mutations.

To conclude the proof of the lemma, first note that by part b) above, a path with at most z mutations does not reach the empty network. Moreover, after the last mutation in the path has taken place, eventually all players best respond, and a) implies that the network will contain no links other than those to player 1. Finally, isolated players will always optimally link player 1 given part b). It follows that after at most z mutations, the dynamics eventually reaches the network with 1 as a center. ■

Lemma 10 *The coradius of the star with player* 1 *as the center satisfies*

$$CR(E^1) \leq \max\left\{ \left\lceil \frac{N-3}{2} - \omega \frac{a_1 - 1}{2} \right\rceil, \left\lceil \frac{N-5}{2} \right\rceil \right\}.$$

Proof. Start from a state that is in a limit set different from the empty network. We need to bound the cost of a path from the current state to the periphery-sponsored star with as the center.

Begin by noting the unperturbed dynamics reaches a state where player 1 is not isolated with positive probability. Otherwise, consider the path in which 1 comes next to best respond. If her best response involves maintaining a link, our claim follows. If not, suppose a terminal node comes to best respond. It must be that this terminal node either drops her link or forms a link to player 1, because if she finds any other link beneficial, 1 should have found that beneficial in the previous period as well. In the latter case our claim is proved. Repeating this argument for all terminal nodes gives a positive probability path leading to a state where agents are either isolated or organized in circles, so that there are no more terminal nodes (recall, the one link property holds). In any circle, a player has a weak better response that involves linking a different member of the circle or dropping her link. In either case a new terminal node is created. Repeating this process eventually takes us to a completely isolated network, contradicting the assumption that the current limit set is not the empty network. Hence we can assume without loss of generality that the current state is such that 1 is not isolated.

Now define

$$s = \max\left\{ \left\lceil \frac{N-3}{2} - \omega \frac{a_1 - 1}{2} \right\rceil, \left\lceil \frac{N-5}{2} \right\rceil \right\}$$

and consider the following path involving a sequence of s consecutive mutations. Along this path, players who mutate drop their current link if they have one, and instead maintain a new link to player 1. Assume that any player already directly connected to 1 does not mutate. Any path with these properties will be sufficient.

After the s mutations, consider an agent j who is linked to by 1. Suppose that j best responds next. If j chooses to maintain no links, her effective neighbors will number at least $a_1(1+\omega)+s$. The reason is that she gains $a_1(1+\omega)$ from player 1, and at least s from 1's direct neighbors. If she decides to maintain a link, she can gain access to at most $N-s-2$ players; the ones she does not access yet. Moreover, her direct neighbors can increase by at most one. Thus the additional gain in effective neighbors from maintaining a link is at most $a_1+N-s-3$. By strong decreasing returns, maintaining a link will not be beneficial for i if $a_1(1+\omega)+s > a_1+N-s-3$, which is satisfied. Therefore the unperturbed dynamics takes us to a state where the player linked by 1 maintains no links.

Next consider an agent j who is either a terminal node or isolated, and not directly connected to player 1. (We will later explore what happens when such an agent does not exist). Suppose that player j best responds next. If j chooses to link 1, her effective number of neighbors will be at least $a_1(1+\omega)+s+1$.

Assume j chooses to link some player h instead. If h is a direct neighbor of 1, then no direct neighbors of h are direct neighbors of 1. The reason is that any such relationship would involve some player maintaining two links, or if $h=i$, then i maintaining at least one link. Neither of these is possible. If h is not a direct neighbor of 1, then the direct neighborhood of h and 1 can have at most one common agent (otherwise, again, some player would maintain more than one link). In either case, a link to h can gain at most $N-s-1$ direct and indirect neighbors to i. This is because she can access at most 2 of 1 and her direct neighbors. This excludes $s+2-2=s$ people, leaving at most $(N-1)-s$ potential neighbors. Therefore the maximum number of effective neighbors j can have is $a_1+(N-s-2+\omega)$, because she may access 1 indirectly.

It follows that if $a_1(1+\omega)+s+1 > a_1+(N-s-2+\omega)$, or equivalently

$$s > \frac{N-3}{2} - \frac{\omega(a_1-1)}{2}$$

holds, j will either choose to link player 1, or decide to maintain no links. But this inequality is satisfied by the definition of s.

As long as there are terminal or isolated players not directly connected to 1, we can repeat the above argument. When there are no more such players, it has to be that all players outside of the direct neighborhood of 1 are either isolated, or are elements of directed circles. In any circle, there is a player who is has a response weakly better than maintaining her current link; and following such an update a new terminal player will be created. We can now repeat the above argument for terminal nodes. In the end, all players not connected directly to 1 have to be isolated. Given that

$f(1+s) > c$ holds by assumption, all isolated players will optimally link player 1. Finally, when all players except for i maintain a link to 1, we can allow 1 to drop her link and let i update next. This results in a periphery-sponsored star with as the center.

To conclude the argument, start from the limit set that is the empty network. After s players mutate and link player 1, all others will have a unique best response that involves doing the same. ■

Under the assumptions of the theorem, for N odd the two lemmas imply that for the star where 1 is the center, the coradius is at most $(N-3)/2$ whereas the radius is at least $1+(N-3)/2$. Invoking the radius-coradius theorem of Ellison immediately gives the result. If N is even, then for $\omega(a_1-1) > 1$ the coradius is at most $\lfloor (N-3)/2 \rfloor$ whereas the radius is at least $\lceil (N-3)/2 \rceil$ and again we have the result.

Proof of Theorem 5

(i) Using the terminology that i maintains a link to j if i pays at least $c/2$ for that link, we claim that no player maintains more than one link. Otherwise, for some link that i maintains, she gets at most half of her total effective neighbors through that link, while she is paying $c/2$ or more. Then dropping this link (setting t_j^i sufficiently negative) has a benefit of at least $c/2$ and a loss of at most $f(n) - f(n/2)$ where n is the total number of i's effective neighbors. Under $(1, c/2)$ strong decreasing returns, dropping the link will be optimal. As a result, any Nash equilibrium will be a directed graph with no player initiating more than one link, and by Lemma 3, each component will have the directed tree and circle architecture.

With $(1, c/2)$ strong decreasing returns, an equilibrium network never contains a circle by the argument used in the proof of Lemma 6. We turn to establish that all terminal nodes maintain a link to the same player. Suppose that there are two terminal nodes, 1 and 2, who link different players i_1 and i_2. Assume that i_1 has access to m_1 effective neighbors, and provides access of g_1 effective neighbors to 1, where g_1 includes the benefit that 1 gets from knowing i_1 as well. Because $a_{\delta+1}/a_\delta$ is weakly decreasing, the inequality $g_1 - a_1 \leq m_1/a_1$ must hold. This is because the $g_1 - a_1$ effective neighbors that 1 has access to through i_1 must all be neighbors of i_1; but then their total value cannot exceed the total that accumulates to i_1 herself (m_1), discounted by the highest relative discount rate $a_2/a_1 = 1/a_1$.

It is easy to see that the surplus over the link between 1 and i_1 is

$$f(g_1) + f(m_1 + a_1) - f(m_1) - c$$

where the first term is the incremental benefit of player 1 for this link, while the second and third terms amount to the incremental benefit of player i_1. Suppose that the surplus of the link between

2 and i_2 exceeds that over the link between 1 and i_1:

$$f(g_1) + f(m_1 + a_1) - f(m_1) \leq f(g_2) + f(m_2 + a_1) - f(m_2). \tag{7}$$

Now consider the deviation where 1 drops her link to i_1 and instead makes an offer to i_2 with a transfer amount chosen such that the payoff of 1 after this move will increase slightly. If it is in $2's$ best interest to accept this offer, then the previous allocation could not have been a pairwise equilibrium with unilateral status quo. The way to prove that 2 will accept the offer is to compute the surplus from the proposed link, and compare it with the surplus of the prior arrangement. Importantly, in this calculation we also need to account for the fact following the deviation, 1 will no longer be linked to i_1. This matters for the change in the payoff of i_2, who was potentially indirectly connected through i_1 to 1 previously. A lower bound for the surplus of the proposed new link between 1 and i_2 is

$$f(g_2 + a_2 - a_{\bar{\delta}}) + f(m_2 + 2a_1 - a_{\bar{\delta}}) - f(m_2 + a_1 - a_{\bar{\delta}}) - c.$$

The first term here is a bound for the benefit accumulating to player 1, accounting for the fact that 1 is no longer connected to i_1, which potentially reduces g_2 by at most a_3. The second and third terms measure the change in the benefits to i_2, who is now directly connected to 1, but lost the indirect connection, which costs her at most a_2. We need this bound to be larger than the surplus over the pre-existing link between 1 and i_1 for a profitable deviation to exist. Because, by (7), the surplus at location 2 was larger originally, showing

$$f(g_2 + a_2 - a_{\bar{\delta}+1}) + f(m_2 + 2a_1 - a_{\bar{\delta}}) - f(m_2 + a_1 - a_{\bar{\delta}}) > f(g_2) + f(m_2 + a_1) - f(m_2)$$

would be sufficient (note, we used the fact that $a_2 = 1$). This inequality becomes harder to satisfy if we increase g_2, because the left hand side grows slower in g_2 than the right hand side Since $g_2 \leq a_1 + m_2/a_1$, we can increase g_2 only up to this point. Introducing $n = 1 + m_2/a_1$ and substituting in $g_2 = n - 1 + a_1$ gives

$$f(n + a_1 - a_{\bar{\delta}+1}) + f((n+1)a_1 - a_{\bar{\delta}}) - f(na_1 - a_{\bar{\delta}}) > f(n - 1 + a_1) + f(na_1) - f((n-1)a_1).$$

To see why this inequality holds, first note that by assumption, $a_1 a_{\bar{\delta}+1} \leq a_{\bar{\delta}}$ and hence

$$f((n+1)a_1 - a_{\bar{\delta}}) - f(na_1 - a_{\bar{\delta}}) \geq f((n+1)a_1 - a_1 a_{\bar{\delta}+1}) - f((na_1 - a_1 a_{\bar{\delta}+1}).$$

Then we can write

$$
\begin{aligned}
f(n + a_1 - a_{\bar{\delta}+1}) + f((n+1)a_1 - a_{\bar{\delta}}) - f(na_1 - a_{\bar{\delta}}) \quad &\geq \\
f(n + a_1 - a_{\bar{\delta}+1}) + f((n+1)a_1 - a_1 a_{\bar{\delta}+1}) - f((na_1 - a_1 a_{\bar{\delta}+1}) \quad &> \quad f(n-1+a_1) + f(na_1) - f((n-1)a_1)
\end{aligned}
$$

where the last step uses (MS). This gives the desired inequality, verifying that all terminal nodes connect to the same player. Since circles are ruled out, this result also shows that any nonempty equilibrium consists of a single component.

Given that the network is a tree with all terminal nodes connecting the same player i, the only remaining possibility to rule out is that i maintains an additional link to some other player, which initiates a chain of agents connecting to one another. This "forward chain" cannot be too long, however. If it has a length of $2d$ or more, then there is a player in the middle of the chain whose indirect neighborhood only includes a line of d players going forward as well as a line of d players going backward. By strong decreasing returns, this agent chooses to drop her link. As a result, the chain can be at most $2d-1$ long, and the remaining $N-2d+1$ or more players are all organized in a star. But then i, the center of this star, will certainly drop her link as long as $2da_1 < (N-2d)a_1$ or equivalently $N > 4d$, which holds by assumption. Hence the network must be a single star.

(ii) We need to alter the terminal node argument in the proof of (i). Consider the same situation as above with two terminal nodes, 1 and 2, connecting to players i_1 and i_2, but now assume that i_1 is not even indirectly connected to 1, thus the distance between i_1 and i_2 is at least $d = 2$. Under these assumptions, we do not need to worry about changes in indirect payoffs. Therefore both equilibrium refinements are equivalent in terms of evaluating the benefit of the proposed deviation by player 1. Since now we also have $d = 2$, the monotonicity of the relative discount rates $a_{\delta+1}/a_\delta$ is automatic, and the above argument shows that any two players who have "incoming" terminal nodes must be within distance 1 from each other.

Given the tree structure of the graph, it must be that there are at most 2 players on a chain who have incoming terminal nodes, and then the chain might continue further. As above, the forward piece of the chain cannot be of length more than $2d-1 = 3$, so there must be at least $N-3$ players who are organized in the two interlinked stars that contain all terminal nodes. The center of the star that is closer to the forward chain will surely drop her link if $2da_1 < (N-2d)a_d$ or if $N > (2 + 2a_1/a_d)d$, which holds by assumption. It follows that the network is either a single star or two interlinked stars directly connected through their centers.