Efficient Division When Preferences are Private: Using the Expected Externality Mechanism

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Abstract

We study the problem of allocating multiple items to two agents whose cardinal preferences are private information. If money is available, Bayesian incentive compatibility and ex-ante Pareto efficiency can be achieved using the Expected Externality Mechanism (EEM). Absent money, under certain reasonable conditions, Bayesian incentive compatibility and ex-post Pareto efficiency remain achievable with a modified EEM that uses one good as a numeraire in lieu of money. We study this modified EEM's properties and compare it with other allocation procedures.

Keywords: Expected Externality Mechanism, Object Allocation, Fair Division
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1 Introduction

The classic problem of consumption allocation, how to divide a set of goods among individuals, has been studied in a variety of settings. The goods may be homogeneous or heterogenous, divisible or indivisible, complements or substitutes; the individuals' preferences may be ordinal or cardinal; and monetary transfers may or may not be feasible. We study the problem of allocating a set of n goods to two individuals who have cardinal preferences that are private information. We consider the case where money is available as a medium of exchange, but focus on the more difficult case when it is not. Our results apply to both divisible and indivisible goods.

Part of the motivation for this paper is the observation that many entities simply opt to allocate an equal amount of each good to each agent, when some variant of a fair division approach would greatly expand achievable valuations. Thus, a university department may give each new assistant professor the same travel budget, the same computer, and the same quantity of research assistance. Ex-post trade in these items is atypical, if not impossible. Similarly, many Internet-based applications allocate multiple resources for their users—such as CPU time and storage capacity—through the use of "one-size-fits-all" constraints. Clearly, users may value different assortments of goods and services differently. A fair division approach, which gave each agent a preferred mix of goods or usage in these two examples, could improve outcomes for all.

When dividing a set of goods between individuals, the usual goals are efficiency and fairness. A challenge is that individuals may report their preferences strategically. A variety of simple methods, such as divide-and-choose (where one person divides the set in two parts and the other person chooses his preferred part) or strict alternation (where individuals take turns in selecting a good from the set that has not yet been claimed), fall short of efficiency. The adjusted winner mechanism (Brams and Taylor, 1999a,b) has several desirable properties assuming truthful reporting. However, when agents have prior beliefs about their counterpart's preferences, strategic misreporting must be expected, since it will offer gains in expected value.

In this paper, we argue that many challenging division problem can be successfully resolved through the use the Expected Externality Mechanism (EEM) (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979).[1] When money is available and unlimited transfers are possible, the EEM is easy to apply, even in difficult division problems. The EEM employs direct revelation. Agents report their private information and these reports enable an efficient assignment of goods. To ensure honest reporting, each agent receives a payment equal to the expected externality due to his report, assuming that the other party reports truthfully. Given such transfers, each agent wants to report his private information truthfully, since he now internalizes the expected aggregate consequences of his report.[2] Bayesian incentive compatibility, budget balance, and ex-ante Pareto efficiency can be achieved. Of course, the latter implies ex-post Pareto efficiency.

In many important situations, money is either not available or not acceptable as a medium of exchange. That is true of pure barter, quite possibly for the division of an estate or a cake (Steinhaus, 1948), and for the division of tasks within a friendship or marriage. Many organizations, such as government bureaus, allocate tasks without using money. Dissolution processes in partnerships, particularly for real property, may give assets in kind, and avoid monetary payments. Fair division processes have the potential to improve outcomes within many of these contexts of dividing up assets or tasks in a household or organization.

If money is unavailable to serve as a numeraire, the challenge is greater. Ex-ante efficiency is lost as the elicitation of cardinal utilities is now impossible. However, despite the absence

[1]See also Chatterjee et al. (1978) and Pratt and Zeckhauser (1981, 1987). A textbook exposition is provided by Mas-Colell et al. (1995, pp. 885–7).

[2]As explained by Tideman and Plassmann (2017), a common thread among efficient mechanisms is that all parties bear the marginal social costs of their actions. This is particularly true of descendants, like the EEM, of the mechanisms proposed by Vickrey (1961), Clarke (1971), and Groves (1973).

of money, under certain reasonable conditions it remains possible to employ a modified EEM and still achieves fair and ex-post efficient outcomes. In this mechanism, one of the goods serves as the numeraire commodity in lieu of money and allocations are determined on the basis of ratios of valuations. We identify necessary and sufficient conditions for such a mechanism to be feasible. The mechanism is incentive compatible and the allocations it specifies are ex-post Pareto efficient. We provide examples demonstrating its advantages over other common allocation schemes, such as equal division, alternating selection, or divide and choose. Though the majority of our analysis focuses on the case of linear utilities, the typical assumption in the literature, the modified EEM can also be applied to division problems with nonlinear utilities or where agents value bundles of goods. We also tackle practical issues related to our mechanism's implementation, including the selection of the appropriate numeraire commodity—surprisingly, it is uniquely determined in many cases.

The remainder of the paper is organized as follows. The next section briefly review the related literature. The problem is formulated in section 3. To establish a benchmark, section 4 introduces the EEM allowing for unlimited monetary transfers. In section 5, we introduce a natural variant of the EEM in a setting without money, and we examine its properties. Section 6 presents brief but important extensions of our model, including the case of more than two agents and nonlinear utility functions. Section 7 concludes.

2 Related Literature

The EEM was introduced by Arrow (1979) and d'Aspremont and Gérard-Varet (1979) in the context of a public goods problem. Chatterjee et al. (1978) apply the idea to a situation involving buyer-seller bargaining. Pratt and Zeckhauser (1981, 1987) discuss the EEM's application to dynamic and multi-stage problems, among other extensions. Further studies of the EEM include Maskin and Laffont (1979) and Krishna and Perry (2000). Each of these preceding studies presumes a classic quasilinear environment where unlimited monetary transfers are feasible. We relax this requirement in our study.

A considerable literature examines the assignment of goods to agents. One strand of this literature focuses on single-unit assignment problems, that is, every agent is allocated one object, possibly probabilistically (e.g., see Koopmans and Beckmann, 1957; Hylland and Zeckhauser, 1979; Leonard, 1983; Zhou, 1990). Ordinal mechanisms, such as the (random) serial dictatorship (Abdulkadiroğlu and Sönmez, 1998) or the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001), have received particular attention. Agents in our model have cardinal utilities. Those utilities play a key role in defining the requirements for efficiency, the allocation rule, and the mechanism we investigate. Kojima (2009) and Budish et al. (2013) allow multiple items to be allocated to a single agent, but the number of items allocated to a given individual is predetermined. In contrast, in our setting that number arises endogenously.

In the most closely-related paper, Miralles (2012) solves for the expected welfare-maximizing mechanism when there are two indivisible objects that are to be allocated among many ex-ante symmetric agents with additive preferences. His model does not allow for monetary transfers. Rather, he employs probability shares of one object as a numeraire commodity. We consider the problem of allocating $n \geq 2$ items, which may have more than unit supply, to two agents.[3] Our analysis allows for asymmetries among agents and for correlation in the items' values to an agent. Unlike Miralles (2012), we do not aim to solve for the optimal mechanism. Instead, we argue that the EEM performs well relative to other appealing allocation procedures.[4] Given its familiarity and appealing properties in standard problems, we believe the extended EEM introduced here provides a practical benchmark for this expanded class of allocation problems.

[3]We sketch our model's extension to more than two agents in section 6.

[4]Miralles (2012, p. 196) notes that extensions of his mechanism to asymmetric or $n > 2$ good environments are not obvious. Instead, he suggests simple mechanisms may be preferable in these more complex scenarios.

By focusing on a simple and well-known mechanism for the division or assignment of goods we follow McAfee (1992). McAfee (1992) investigates several "simple mechanisms" for partnership dissolution.[5] His analysis of the alternating selection mechanism (ASM), where two parties take turns selecting available items, is of particular interest. We show by example that the EEM outperforms the ASM in our setting by achieving greater expected aggregate welfare.

3 Model

There are two agents, A and B, and n goods. Denote by $s_k \in \mathbb{R}_+$ the available quantity of good k and let $s := (s_1, \ldots, s_n)$. Denote the valuation of agent $i \in \{A, B\}$ per unit of good k by $v_{ik} \in \mathbb{R}_+$ and let $v_i := (v_{i1}, \ldots, v_{in})$. Denote the profile of agents' valuations by $v := (v_A, v_B)$. The agents' preferences across goods are additively separable and linear in any monetary payments. If $q_i := (q_{i1}, \ldots, q_{in})$ is the vector of assigned quantities[6] of each good to agent i and t_i is the (net) monetary transfer received by agent i, then the agent's utility is

$$u_i(q_i, t_i | v_i) = v_i \cdot q_i + t_i.$$

In section 5 we focus on the case where monetary transfers are not feasible. In that case, we set $t_i = 0$ for all i and an agent's utility is

$$u_i(q_i | v_i) = v_i \cdot q_i.$$

[5]The problem of partnership dissolution has also been investigated by Cramton et al. (1987). They consider the allocation of one item among many agents, each of whom may already own a fractional stake in the item. Allowing for monetary transfers, they characterize the mechanisms achieving efficient dissolution. Efficient dissolution involves allocating the item in its entirety to the shareholder who values the item the most. We do not presume ex-ante fractional ownership of any of the items to be allocated.

[6]Our analysis and results also apply to the case where each unit of each good is indivisible if potentially probabilistic assignments are allowed. In that case, if q_{ik} is an integer, it represents the number of units of good k that are (deterministically) assigned to agent i. If q_{ik} is not an integer, agent i gets $\lfloor q_{ik} \rfloor$ units of good k with probability 1; agent i then gets an additional unit of good k with probability $q_{ik} - \lfloor q_{ik} \rfloor$.

The valuations of the two agents are independently drawn from the distributions F_A and F_B, which are common knowledge. The valuations of particular goods to an agent may be correlated. Each agent observes his valuations, but not the valuations of the other agent. Let $\mathscr{V}_i$ be the support of the valuation distribution for agent i. We write $\tilde{v}_i := (\tilde{v}_{i1}, \ldots, \tilde{v}_{in})$ to denote agent i's valuation vector as a random variable. As usual, $\mathbb{E}[\cdot]$ denotes the expected value operator and $\mathbb{E}_{\tilde{x}}[\cdot]$ means that expectations are taken with respect to only a particular random variable $\tilde{x}$. Throughout our analysis we assume all expectations exist.

An allocation rule γ is a mapping from agents' valuations to assignments of goods and transfers (if applicable).[7] Let $\gamma_i(v)$ be the assignment of agent i at valuation profile v. We are interested in mechanisms that yield efficient allocations. Two concepts of welfare and efficiency arise in our analysis.[8]

Given γ, agent i's *ex-post utility* at $v = (v_i, v_j)$ is $u_i(\gamma_i(v)|v_i)$. A rule γ is *ex-post Pareto efficient* if no alternative rule γ' exists such that $u_i(\gamma_i'(v)|v_i) \geq u_i(\gamma_i(v)|v_i)$ for all i and all $v = (v_i, v_j)$, with at least one inequality being strict. When a rule is ex-post Pareto efficient, no mutually beneficial trades are available after agents learn their valuations and receive their allocations and transfers.

Agent i's *ex-ante utility* is $U_i(\gamma) = \mathbb{E}[u_i(\gamma_i(\tilde{v})|\tilde{v}_i)]$. A rule γ is *ex-ante Pareto efficient* if no alternative rule γ' exists such that $U_i(\gamma') \geq U_i(\gamma)$ for all i with at least one inequality being strict. It is well known that γ is ex-ante Pareto efficient if and only if it maximizes the social welfare function

$$W(\gamma) = \sum_i \lambda_i \mathbb{E}\left[u_i(\gamma_i(\tilde{v})|\tilde{v}_i)\right] \tag{1}$$

for some set $\{\lambda_i\}$ of nonnegative utility weights. Ex-ante Pareto efficiency implies ex-post Pareto efficiency.

[7] A mechanism may also depend on the particulars of the environment, such as the valuation distributions, through we suppress this dependence in our notation.

[8] Our exposition of these welfare criteria follows Holmström and Myerson (1983). Our definitions correspond to their "classical" notions ignoring incentive compatibility.

Valuations are private information and agents must report them to the mechanism to determine allocations. Let $\hat{v}_i \in \mathscr{V}_i$ be a report submitted by agent i. We focus on mechanisms that are Bayesian incentive compatible.[9] A mechanism γ is *Bayesian incentive compatible* if it is a Bayesian Nash equilibrium for each agent to truthfully report his valuations.[10] That is, for each $i, j \in \{A, B\}$, $i \neq j$, and $v_i \in \mathscr{V}_i$, $\mathbb{E}[u_i(\gamma_i(v_i, \tilde{v}_j)|v_i)] \geq \mathbb{E}[u_i(\gamma_i(\hat{v}_i, \tilde{v}_j)|v_i)]$ for all $\hat{v}_i \in \mathscr{V}_i$.

4 The Expected Externality Mechanism Given Money

As a benchmark, consider first the case where money is available as a medium of exchange. In this case, it is possible to achieve an ex-ante Pareto efficient allocation using the EEM (d'Aspremont and Gérard-Varet, 1979; Arrow, 1979). The EEM is a direct mechanism where agents report their values. It consists of an allocation rule $q(\hat{v}) := (q_A(\hat{v}), q_B(\hat{v}))$ and a transfer function $t(\hat{v}) := (t_A(\hat{v}), t_B(\hat{v}))$. Given the agents' reports $\hat{v}_A$ and $\hat{v}_B$, the allocation of each good $k = 1, \ldots, n$ is determined as follows:

$$q_{Ak}(\hat{v}_A, \hat{v}_B) = \begin{cases} s_k & \text{if } \hat{v}_{Ak} > \hat{v}_{Bk} \\ s_k/2 & \text{if } \hat{v}_{Ak} = \hat{v}_{Bk} \\ 0 & \text{if } \hat{v}_{Ak} < \hat{v}_{Bk} \end{cases} \qquad q_{Bk}(\hat{v}_A, \hat{v}_B) = \begin{cases} 0 & \text{if } \hat{v}_{Ak} > \hat{v}_{Bk} \\ s_k/2 & \text{if } \hat{v}_{Ak} = \hat{v}_{Bk} \\ s_k & \text{if } \hat{v}_{Ak} < \hat{v}_{Bk} \end{cases}. \tag{2a}$$

[9] Strategy-proofness is a stronger incentive criterion. A direct mechanism is *strategy-proof* if it is always optimal for an agent to truthfully report his valuations regardless of the other agent's strategy. Requiring an allocation mechanism to be strategy-proof readily leads to several known impossibility results in our setting (Green and Laffont, 1977; Krishna and Perry, 2000).

[10] Though we maintain a Bayesian perspective throughout our study, Gorelkina (2018) has recently examined the EEM's properties in a setting characterized by iterative thinking. She shows that the EEM can be adjusted to a "Level-k" framework while maintaining many of its key properties.

In words, if the two agents report the same valuation for good k, the s_k units of good k are split evenly.[11] Otherwise, the agent reporting the highest valuation receives all s_k units of good k.

Given (2a), the (net) transfer received by agent A in the EEM is

$$t_A(\hat{v}_A, \hat{v}_B) = \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_B(\hat{v}_A, \tilde{v}_B)\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_A(\tilde{v}_A, \hat{v}_B)\right]. \tag{2b}$$

The (net) transfer received by agent B is defined symmetrically. In fact, $t_B(\hat{v}_A, \hat{v}_B) = -t_A(\hat{v}_A, \hat{v}_B)$. Thus, the EEM's (net) transfers sum to zero for all possible reports. In other words, the mechanism satisfies *budget balance*.

The transfer function (2b) has two parts. The first component, $\mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_B(\hat{v}_A, \tilde{v}_B)\right]$, is the expected surplus accruing to agent B as a function of A's announced valuation, $\hat{v}_A$. It is the *expected externality* associated with A's report since it is the component of expected aggregate welfare that depends on $\hat{v}_A$ but is not appropriated by A himself. The second component, $\mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_A(\tilde{v}_A, \hat{v}_B)\right]$, does not depend on A's announcement and ensures that transfers balance. It is the expected externality associated with agent B's announced valuation.

Given (2a), (2b), and the conjecture that agent B truthfully announces his valuation ($\hat{v}_B = v_B$), agent A will announce $\hat{v}_A$ to maximize $\mathbb{E}_{\tilde{v}_B}[v_A \cdot q_A(\hat{v}_A, \tilde{v}_B) + t_A(\hat{v}_A, \tilde{v}_B)]$, which equals

$$\underbrace{\mathbb{E}_{\tilde{v}_B}\left[v_A \cdot q_A(\hat{v}_A, \tilde{v}_B)\right]}_{\substack{\text{Expected surplus} \\ \text{appropriated by A}}} + \overbrace{\underbrace{\mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_B(\hat{v}_A, \tilde{v}_B)\right]}_{\substack{\text{Expected externality} \\ \text{given A's report}}} - \mathbb{E}\left[\tilde{v}_A \cdot q_A(\tilde{v}_A, \tilde{v}_B)\right]}^{\mathbb{E}_{\tilde{v}_B}[t_A(\hat{v}_A, \tilde{v}_B)]}. \tag{3}$$

Since the final term is independent of $\hat{v}_A$, the maximizer of (3) will maximize expected aggregate surplus conditional on v_A, the sum of the first two terms in (3).

[11] We assume equal division for fairness and simplicity. However, any specified division of a good that the agents value the same will be efficient.

Proposition 1 is a standard result concerning the EEM. See any of the above-cited papers or the textbook Mas-Colell et al. (1995, pp. 885–7). We omit the proof for brevity.

Proposition 1. *When money is available as a medium of exchange, the EEM is Bayesian incentive compatible and ex-ante Pareto efficient. Moreover, the EEM maximizes the social welfare function* (1) *where all agents are afforded equal weight.*

While (2b) is the typical specification of transfers in the EEM, other formulas achieve essentially the same result in our setting. One interesting case recasts transfers as compensation paid for expected harms that an agent's report imposes on the other. Specifically, after announcing $\hat{v}_A$, agent A pays $\mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_A(\hat{v}_A, \tilde{v}_B)\right]$ directly to agent B. This amount represents the expected cost that A's report imposes on B given that B is truthfully reporting his valuations. Whenever $v_{Bk} < \hat{v}_{Ak}$, agent B is prevented from enjoying a benefit of $v_{Bk}s_k$ because s_k units of good k are taken by agent A. Symmetrically, agent B pays $\mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_B(\tilde{v}_A, \hat{v}_B)\right]$ to agent A as compensation. Given the bilateral payments, the net transfer to agent A is

$$t'_A(\hat{v}_A, \hat{v}_B) = \underbrace{\mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_B(\tilde{v}_A, \hat{v}_B)\right]}_{\text{Payment from B to A}} - \underbrace{\mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_A(\hat{v}_A, \tilde{v}_B)\right]}_{\text{Payment from A to B}}. \tag{4}$$

The net transfer to agent B, $t'_B(\hat{v}_A, \hat{v}_B)$, is defined symmetrically.

Proposition 2. *When money is available as a medium of exchange, the EEM where (net) transfers are instead defined by* (4) *is Bayesian incentive compatible and ex-ante Pareto efficient.*

Proof. It is sufficient to show that $t_A(\hat{v}_A, \hat{v}_B)$ and $t'_A(\hat{v}_A, \hat{v}_B)$ differ by a constant. From (2a),

$q_{Ak}(\hat{v}_A, \hat{v}_B) + q_{Bk}(\hat{v}_A, \hat{v}_B) = s_k$ for all k. Thus,

$$\begin{aligned}
t_A(\hat{v}_A, \hat{v}_B) &= \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_B(\hat{v}_A, \tilde{v}_B)\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_A(\tilde{v}_A, \hat{v}_B)\right] \\
&= \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot (s - q_A(\hat{v}_A, \tilde{v}_B))\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot (s - q_B(\tilde{v}_A, \hat{v}_B))\right] \\
&= \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot s\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot s\right] + \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot q_B(\tilde{v}_A, \hat{v}_B)\right] - \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot q_A(\hat{v}_A, \tilde{v}_B)\right] \\
&= \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot s\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot s\right] + t'_A(\hat{v}_A, \hat{v}_B).
\end{aligned}$$

The value $\mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot s\right] - \mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot s\right]$ is a constant, thus proving the proposition. □

The EEM with the transfer function (4) satisfies both ex-ante and interim individual rationality. In other words, an individual wants to participate both before and after learning his valuation, but before learning the final allocation. In particular, agent i with valuation v_i secures a nonnegative expected payoff by reporting the lowest possible valuation. After a truthful report, his expected utility must be at least as large in equilibrium. In some other applications the EEM may fail to satisfy interim individual rationality (Maskin and Laffont, 1979). The existence of an individually-rational mechanism meeting our desiderata in this allocation problem is implied by Theorem 2 of Krishna and Perry (2000).

The next corollary follows immediately from the proof of Proposition 2 and the preceding discussion. It applies when the agents are ex-ante symmetric and in many cases of asymmetry as well.

Corollary 1. *Suppose* $\mathbb{E}_{\tilde{v}_A}\left[\tilde{v}_A \cdot s\right] = \mathbb{E}_{\tilde{v}_B}\left[\tilde{v}_B \cdot s\right]$.

(a) The transfer rules (2b) *and* (4) *are equivalent.*

(b) The EEM with transfer rule (2b) *satisfies ex-ante and interim individual rationality for both agents.*

5 The EEM Absent Money

When monetary transfers are feasible, the EEM is ex-ante Pareto efficient and maximizes (1) assuming each agent is afforded equal weight, $\lambda_A = \lambda_B$. Maximizing (1) in an incentive compatible way when money is not available is generally impossible, except in the dictatorial case.[12] As an example, suppose the valuation of each good k to each agent i is continuously distributed on the interval $[\underline{v}_k, \overline{v}_k]$ where $\underline{v}_k < \overline{v}_k$. To maximize (1) when $\lambda_A = \lambda_B$ requires assigning each good to the person who values it most. Therefore, in a direct mechanism each good (in its entirety) would be allocated to the agent reporting the highest value. Now, however, each agent would use the strategy that maximizes his probability of getting each good—both agents would announce a valuation of $\overline{v}_k$ for every k. The mechanism would have no way to determine to whom the good should be allocated. Thus, any mechanism would have to risk giving a good (or part of it) to an individual who values it less.

Acknowledging the limits when ex-ante Pareto efficiency is imposed as a constraint, we focus instead on arriving at an ex-post Pareto efficient assignment of goods.

Remark 1. If the agents' valuations are known, the set of ex-post Pareto efficient allocations of goods $k = 1, \ldots, n$ can be found in the following way. Choose some fixed threshold θ. If $v_{Ak}/v_{Bk} > \theta$, then good k is allocated to agent A in its entirety. If $v_{Ak}/v_{Bk} < \theta$, then good k is allocated to agent B in its entirety. If $v_{Ak}/v_{Bk} = \theta$, then good k is split between the two agents.[13] By varying θ we arrive at a family of efficient allocations. It is immediate from (2a) that the EEM allocation corresponds to the $\theta = 1$ case.

Despite the absence of money, a natural modification of the EEM can be used to allocate goods in an incentive compatible and ex-post efficient manner, under certain reasonable conditions. This modification of the EEM uses relative valuations with respect to an exoge-

[12] Suppose only agent A is afforded non-zero weight in (1). In this case, the mechanism that always allocates everything to agent A is ex-ante Pareto efficient and incentive compatible. Such a mechanism is neither fair nor particularly interesting to study.

[13] Any split of the good yields an ex-post efficient assignment.

nously set numeraire good to determine allocations. Appropriate transfers of the selected numeraire then assure honest reporting. Below we define this extension of the EEM and we investigate its properties.

Suppose good m is chosen as the numeraire and consider the following allocation procedure. Ask the agents to report their valuations for each good. Let $\hat{r}_{ik} = \hat{v}_{ik}/\hat{v}_{im}$ be the ratio of agent i's reported valuation for good k over his reported valuation for good m. For each $k \neq m$, implement the allocation function

$$q_{Ak}(\hat{v}_A, \hat{v}_B) = \begin{cases} s_k & \text{if } \hat{r}_{Ak} > \hat{r}_{Bk} \\ s_k/2 & \text{if } \hat{r}_{Ak} = \hat{r}_{Bk} \\ 0 & \text{if } \hat{r}_{Ak} < \hat{r}_{Bk} \end{cases} \qquad q_{Bk}(\hat{v}_A, \hat{v}_B) = \begin{cases} 0 & \text{if } \hat{r}_{Ak} > \hat{r}_{Bk} \\ s_k/2 & \text{if } \hat{r}_{Ak} = \hat{r}_{Bk} \\ s_k & \text{if } \hat{r}_{Ak} < \hat{r}_{Bk} \end{cases} \tag{5a}$$

The allocation function (5a) assigns the entirety of good k to the agent reporting the highest valuation ratio. If both agents report the same ratio, the good is split evenly.

Next, define EEM-style transfers to incentivize truthful reporting. Instead of transfers of money, the mechanism identifies quantities of good m that each agent gives to the other agent starting from an "endowment" of $s_m/2$ units. Thus, the allocation of good m to agent A is

$$q_{Am}(\hat{v}_A, \hat{v}_B) = \frac{s_m}{2} + \mathbb{E}_{\tilde{v}_B}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\hat{v}_A, \tilde{v}_B)\right] - \mathbb{E}_{\tilde{v}_A}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \hat{v}_B)\right] \tag{5b}$$

and the allocation to B, $q_{Bm}(\hat{v}_A, \hat{v}_B)$, is defined symmetrically. Expression (5b) is directly analogous to the transfer function (2b). The first summation is the expected (relative) value of the goods $k \neq m$ received by B given A's announcement. The second summation is the expected (relative) value of the goods $k \neq m$ received by A given B's announcement. We call the mechanism defined by (5a) and (5b) the *EEM with respect to numeraire good m*, or the EEM(m).

The EEM(m) is *feasible* whenever (5b) defines a valid assignment of good m. That is, $q_{Am} \geq 0$ and $q_{Bm} \geq 0$ for all possible values of $\hat{v}_A$ and $\hat{v}_B$. This ensures that there is sufficient quantity of the chosen numeraire for the compensatory transfers. The following lemma identifies a sufficient and necessary condition for feasibility. Examples 1 and 2, presented after Proposition 4, demonstrate instances of feasible division problems with discrete and continuous valuations.

Lemma 1. *Fix m and for each $i \in \{A, B\}$ and $j \neq i$ let*

$$\overline{m}_i := \sup_{\hat{v}_i \in \mathscr{V}_i} \mathbb{E}_{\tilde{v}_j}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{jk}}{\tilde{v}_{jm}}\right) q_{jk}(\hat{v}_i, \tilde{v}_j)\right] \quad \textit{and} \quad \underline{m}_i := \inf_{\hat{v}_i \in \mathscr{V}_i} \mathbb{E}_{\tilde{v}_j}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{jk}}{\tilde{v}_{jm}}\right) q_{jk}(\hat{v}_i, \tilde{v}_j)\right]$$

be the maximum and minimum, respectively, values of the expected (relative) externality resulting from agent i's report. The EEM(m) is feasible if and only if

$$\max\{\overline{m}_A - \underline{m}_B, \overline{m}_B - \underline{m}_A\} \leq \frac{s_m}{2}. \tag{6}$$

Proof. Feasibility requires that $0 \leq q_{im}(\hat{v}_A, \hat{v}_B) \leq s_m$ for all $(\hat{v}_A, \hat{v}_B)$. Noting that $q_{jm}(\hat{v}_A, \hat{v}_B) = s_m - q_{im}(\hat{v}_A, \hat{v}_B)$, applying the definition (5b), and taking the appropriate supremum and infimum gives the desired conclusion. □

Remark 2. It is possible to slightly strengthen Lemma 1 by modifying (5b) and replacing (6) with

$$\overline{m}_A - \underline{m}_B + \overline{m}_B - \underline{m}_A \leq s_m. \tag{6$'$}$$

When (6′) holds, we can find values x_{Am} and x_{Bm} such that $x_{Am} \geq \overline{m}_A - \underline{m}_B$, $x_{Bm} \geq \overline{m}_B - \underline{m}_A$, and $x_{Am} + x_{Bm} = s_m$. Then, in (5b) we would replace $s_m/2$ by x_{Am}; we would also replace $s_m/2$ by x_{Bm} in the corresponding expression for agent B. Unless noted otherwise, we presume the symmetric specification (5b) of the EEM(m) in subsequent analysis.

Proposition 3. *Suppose that money is not available as a medium of exchange. Fix good m as the chosen numeraire commodity. If the EEM(m) is feasible, then it identifies an ex-post Pareto efficient allocation and is Bayesian incentive compatible.*

Proof. Suppose that agents report their valuations truthfully. Then all goods k for which $v_{Ak}/v_{Bk} > v_{Am}/v_{Bm}$ are allocated to agent A in their entirety; all goods k for which $v_{Ak}/v_{Bk} < v_{Am}/v_{Bm}$ are allocated to agent B in their entirety; and all goods k for which $v_{Ak}/v_{Bk} = v_{Am}/v_{Bm}$ (including good m) are split between the two agents. Such allocations are ex-post Pareto efficient.[14]

We now show that the EEM(m) satisfies Bayesian incentive compatibility. Given his valuations v_A, agent A will report the valuations $\hat{v}_A$ that maximize his expected utility assuming that agent B is reporting his valuations truthfully. As the allocation depends only on the reported valuation ratios, it is sufficient to show that agent A has no incentive to misreport $\hat{v}_{Ak}/\hat{v}_{Am}$ for all $k \neq m$.

Suppose agent B truthfully reveals his type, i.e. $\hat{v}_B = v_B$. Thus, with his report, agent

[14] In reference to Remark 1, the ratio v_{Am}/v_{Bm} assumes the role of the "θ" parameter.

A aims to maximize

$$
\begin{aligned}
&\mathbb{E}_{\tilde{v}_B}\left[v_A \cdot q_A(\hat{v}_A, \tilde{v}_B)\right] \\
&= \mathbb{E}_{\tilde{v}_B}\Bigg[\sum_{k\neq m} v_{Ak} q_{Ak}(\hat{v}_A, \tilde{v}_B) \\
&\qquad + v_{Am}\left(\frac{s_m}{2} + \mathbb{E}_{\tilde{v}_B}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\hat{v}_A, \tilde{v}_B)\right] - \mathbb{E}_{\tilde{v}_A}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right]\right)\Bigg] \\
&= \mathbb{E}_{\tilde{v}_B}\left[\sum_{k\neq m} v_{Ak} q_{Ak}(\hat{v}_A, \tilde{v}_B) + v_{Am}\sum_{k\neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\hat{v}_A, \tilde{v}_B)\right] \\
&\qquad + v_{Am}\left(\frac{s_m}{2} + \mathbb{E}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right]\right) \\
&= \mathbb{E}_{\tilde{v}_B}\left[\sum_{k\neq m} v_{Am} \underbrace{\left(\frac{v_{Ak}}{v_{Am}} q_{Ak}(\hat{v}_A, \tilde{v}_B) + \frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}} q_{Bk}(\hat{v}_A, \tilde{v}_B)\right)}_{(\star)}\right] + \text{terms independent of } \hat{v}_A.
\end{aligned}
$$

Now consider the term indicated with an $(\star)$ for a particular realization of $\tilde{v}_B$. If $v_{Ak}/v_{Am} > \tilde{v}_{Bk}/\tilde{v}_{Bm}$, then $(\star)$ is maximized when $\hat{v}_{Ak}/\hat{v}_{Am} = v_{Ak}/v_{Am}$ since all of good k is then allocated to agent A. Instead, if $v_{Ak}/v_{Am} < \tilde{v}_{Bk}/\tilde{v}_{Bm}$, then $(\star)$ is maximized when $\hat{v}_{Ak}/\hat{v}_{Am} = v_{Ak}/v_{Am}$ since all of good k is then allocated to agent B. If $v_{Ak}/v_{Am} = \tilde{v}_{Bk}/\tilde{v}_{Bm}$, then $\hat{v}_{Ak}/\hat{v}_{Am} = v_{Ak}/v_{Am}$ implies an equal split of good k. In each contingency, truthful revelation of valuations is optimal given that agent B is reporting his valuations truthfully. Similarly, truthful reporting is optimal for agent B, if agent A is truthful. Thus, the EEM(m) satisfies Bayesian incentive compatibility. $\square$

Remark 3. As in the case with monetary transfers, we can recast the transfers of the numeraire commodity as compensation paid for expected harm. In this case, we replace $q_{Am}(\hat{v}_A, \hat{v}_B)$ defined in (5b) with

$$
q'_{Am}(\hat{v}_A, \hat{v}_B) = \frac{s_m}{2} + \mathbb{E}_{\tilde{v}_A}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Bk}(\tilde{v}_A, \hat{v}_B)\right] - \mathbb{E}_{\tilde{v}_B}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Ak}(\hat{v}_A, \tilde{v}_B)\right].
$$

The function $q'_{Bm}(\hat{v}_A, \hat{v}_B)$ is defined analogously. The interpretation of $q'_{Am}(\hat{v}_A, \hat{v}_B)$ parallels that of $t'(\hat{v}_A, \hat{v}_B)$ above. The analogue to Corollary 1 applies too.[15] In particular, if the agents are ex-ante symmetric, then $q_{Am}(\hat{v}_A, \hat{v}_B) = q'_{Am}(\hat{v}_A, \hat{v}_B)$.

The next proposition provides a performance guarantee for the EEM(m).

Proposition 4. *Suppose agents A and B are ex-ante symmetric and that the EEM(m) is feasible. The EEM(m) guarantees an ex-ante utility of* $\mathbb{E}_{\tilde{v}_i}[\tilde{v}_i \cdot s]/2$ *to each agent* $i \in \{A, B\}$.

Proof. Without loss of generality we focus on agent A's payoff. In equilibrium, both agents truthfully report their valuations to the mechanism; hence, agent A's ex-ante payoff is $U_A(q) = \mathbb{E}[\tilde{v}_A \cdot q_A(\tilde{v}_A, \tilde{v}_B)]$. Expanding this expression, substituting (5b), and collecting terms gives

$$U_A(q) = \mathbb{E}\left[\sum_{k \neq m} \tilde{v}_{Ak} q_{Aj}(\tilde{v}_A, \tilde{v}_B) + \tilde{v}_{Am}\frac{s_m}{2}\right] + \mathbb{E}\left[\tilde{v}_{Am}\mathbb{E}_{\tilde{v}_B}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\tilde{v}_A, \tilde{v}_B)\right]\right]$$
$$- \mathbb{E}\left[\tilde{v}_{Am}\mathbb{E}_{\tilde{v}_A}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right]\right].$$

Observe that the random variables $\tilde{v}_{Am}$ and $\mathbb{E}_{\tilde{v}_B}\left[\sum_{k \neq m}(\tilde{v}_{Bk}/\tilde{v}_{Bm})q_{Bk}(\tilde{v}_A, \tilde{v}_B)\right]$ are positively correlated while $\tilde{v}_{Am}$ and $\mathbb{E}_{\tilde{v}_A}\left[\sum_{k \neq m}(\tilde{v}_{Ak}/\tilde{v}_{Am})q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right]$ are independent.[16] Recalling that $\mathbb{E}[XY] \geq \mathbb{E}[X]\mathbb{E}[Y]$ when random variables X and Y are positively correlated, we can

[15] Ex-ante and interim individual rationality are trivially assured in a setting without monetary transfers. Before the mechanism's operation, each agent has nothing and negative allocations or transfers are impossible.

[16] Agents' types are independent and the latter expression is only a function of the random variable $\tilde{v}_B$.

conclude that

$$
\begin{aligned}
U_A(q) \geq \mathbb{E}\left[\sum_{k\neq m} \tilde{v}_{Ak} q_{Ak}(\tilde{v}_A, \tilde{v}_B) + \tilde{v}_{Am}\frac{s_m}{2}\right] &+ \mathbb{E}\left[\tilde{v}_{Am}\right]\mathbb{E}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\tilde{v}_A, \tilde{v}_B)\right] \\
&- \mathbb{E}\left[\tilde{v}_{Am}\right]\mathbb{E}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right] \\
= \mathbb{E}\left[\sum_{k\neq m} \tilde{v}_{Ak} q_{Ak}(\tilde{v}_A, \tilde{v}_B) + \tilde{v}_{Am}\frac{s_m}{2}\right]& \\
\geq \sum_{k\neq m} \mathbb{E}\left[\tilde{v}_{Ak}\right]\mathbb{E}\left[q_{Ak}(\tilde{v}_A, \tilde{v}_B)\right] + \mathbb{E}\left[\tilde{v}_{Am}\right]\frac{s_m}{2}& \\
= \sum_{k\neq m} \mathbb{E}\left[\tilde{v}_{Ak}\right]\frac{s_k}{2} + \mathbb{E}\left[\tilde{v}_{Am}\right]\frac{s_m}{2}.&
\end{aligned}
$$

The first equality follows from the fact that the agents are ex-ante symmetric. The second inequality follows from $\tilde{v}_{Ak}$ and $q_{Ak}(\tilde{v}_A, \tilde{v}_B)$ being positively correlated. The final equality follows from symmetry. Ex-ante each agent expects to get on average $1/2$ of each good $k \neq m$. Of course, the final line equals $\mathbb{E}_{\tilde{v}_i}[\tilde{v}_i \cdot s]/2$. □

The bound in Proposition 4 amounts to an even split of every good. Thus, each agent expects to receive more than half the goods' total value given his private assessment. Though tight, the bound in Proposition 4 is a very conservative estimate of the EEM(m)'s efficacy in all by extreme situations.[17] As clear from the proposition's proof, even the slightest bit of uncertainty in the items' values lets the EEM(m) exceed the "even division" lower bound as demonstrated by the following examples.

Example 1 (Discrete Valuations)**.** There are two agents and two goods with unit supply. For each agent i, $v_i = (6, 2)$ with probability $1/9$, $v_i = (4, 8)$ with probability $4/9$, and $v_i = (1, 10)$ with probability $4/9$. Good two is the numeraire; therefore, the mechanism will

[17]When the valuation distribution is degenerate and both agents value each good exactly the same, the EEM(m) produces an even split of every good. Thus, each agent's expected payoff is exactly $(v_i \cdot s)/2$. Presumably, any sensible allocation procedure would implement the same outcome in this case.

Table 1: EEM(m) allocation (q_A, q_B) in Example 1.

		Announcement of Agent B ($\hat{v}_B$)		
		(6,2)	(4,8)	(1,10)
Announcement of Agent A ($\hat{v}_A$)	(6,2)	$(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$	$(1,\frac{2}{9}),(0,\frac{7}{9})$	$(1,\frac{4}{45}),(0,\frac{41}{45})$
	(4,8)	$(0,\frac{7}{9}),(1,\frac{2}{9})$	$(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$	$(1,\frac{11}{30}),(0,\frac{19}{30})$
	(1,10)	$(0,\frac{41}{45}),(1,\frac{4}{45})$	$(0,\frac{19}{30}),(1,\frac{11}{30})$	$(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$

use the ratios $\hat{v}_{i1}/\hat{v}_{i2}$ of the reported valuations to determine the allocation. It can be easily verified that condition (6) in Lemma 1 is satisfied and the EEM(m) is feasible.

Table 1 summarizes the EEM(m) allocation rule as a function of the agents' announcements. If both agents report the same values, the goods are split evenly. Otherwise, good 1 is allocated in its entirety to the agent reporting the greater relative value. For example, if $\hat{v}_A = (6, 2)$ and $\hat{v}_B = (4, 8)$, agent A is allocated 1 unit of good 1 and 2/9 units of good 2. The remainder of good 2 is allocated to agent B. In equilibrium, the agents report their values truthfully and the resulting allocation is ex-post Pareto efficient. The ex-ante utility of each agent is approximately 6.32.

It is interesting to compare the EEM(m)'s performance in this example with other well-known allocation procedures. An even split of each good yields an ex-ante utility for each agent of approximately 5.56. Per Proposition 4, the EEM(m) is far superior.

The goods in this example could also be allocated by the *alternating selection mechanism* (ASM) (McAfee, 1992). Under this scheme, one agent is randomly selected to pick first and the agents alternate selecting all units of their most-preferred, still-available good. In this example, the ex-ante utility of each agent given the ASM is approximately 6.07. While the ASM improves upon naive equal division in this case, it is inferior to the EEM(m), where the expected payoff was 6.32.[18]

[18]Whereas agents' types in this example are independent draws from the same distribution, the items' values to an agent can be correlated in our model. Such correlation considerably complicates equilibrium behavior in the ASM. When there are two goods, as above, the agent picking first should just take his most-

Finally, consider the *divide and choose mechanism* (DCM). The DCM randomly selects one agent to propose a division of the goods into two bundles. The second agent then selects his preferred bundle. The division is made at the interim stage—the divider knows his own valuations but not those of the chooser. A notable disadvantage of the DCM in a Bayesian setting with multiple divisible goods is that it burdens the divider with an exceptionally-tricky utility maximization problem. Except in knife-edge cases, he will divide the goods so that he is more than 50% likely to receive his preferred bundle. When creating bundles, he will shift goods to his preferred bundle until his loss in likelihood of receiving that bundle just balances the gain in expected value.[19] In this example, the divider's ex-ante utility is approximately 5.87 while the chooser's ex-ante utility is approximately 6.20. If an agent is equally likely to assume either role, his ex-ante utility is approximately 6.03, again significantly less than the EEM(m)'s value of 6.32.

Example 2 (Continuous Valuations)**.** Suppose that for each agent $i \in \{A, B\}$ and for each item $k \in \{1, 2\}$, $\tilde{v}_{ik} \overset{i.i.d.}{\sim} U[1, 2]$. Three units of good 1 are available and one unit of good 2 is available. Using good 1 as the numeraire, the EEM(m) is feasible and gives each agent an ex-ante utility of approximately 3.118. An even split of the goods yields an expected payoff of 3 to each agent. Under the ASM, the agent picking first will claim all three units of good 1 and all of good 2 will go to the other agent. An agent's ex-ante utility is 3, with a considerable advantage afforded to the agent lucky enough to pick first. Finally, the DCM also lags the EEM(m)'s performance, albeit slightly. It yields an ex-ante utility of

preferred item. But when there are three or more goods, the myopic strategy of picking the most preferred available item may not be an equilibrium of the ASM (McAfee, 1992). An agent's choice might signal his valuation for some remaining items. An agent may defer selecting a valuable item in a particular round if the other agent values it differently. The EEM(m)'s incentive properties are unaffected when the items' values to an agent are correlated. Truth-telling remains an equilibrium.

[19]It is often suggested that the divider should adopt a "max-min" strategy where he splits the goods into two parts that he regards as equally desirable. This strategy is not optimal if the divider has beliefs or information concerning the chooser's preferences. For elaboration on the DCM see Crawford (1977) or Brams and Taylor (1999b, Chapter 4).

approximately 3.108 per agent.[20]

To better appreciate the EEM(m)'s operation, it is instructive to compare its performance in this example to that of the EEM when unlimited monetary transfers are allowed. As expected, the EEM with unlimited monetary transfers is superior to the EEM(m) in terms of ex-ante utility. The former gives each agent an ex-ante utility of 3.334 while the latter only provides 3.118. It is interesting to observe that the distributions of ex-post payoffs between mechanisms differ, as illustrated in Figure 1.[21] Payoffs under the EEM(m) are much more concentrated. To understand why, recall that under the EEM(m), an agent with relatively low valuations for both goods, say $v_{i1} \approx 1$ and $v_{i2} \approx 1$, is entitled to receive approximately $s_1/2$ of the numeraire good 1. Thus, his worst case ex-post payoff is 1.5. In contrast, in the EEM with transfers, an agent with those same valuations will not receive either good and the transfer he receives will be trivial. He imposes relatively little externality on the other party who almost surely values the items much more. Thus, the agent's ex-post payoff is close to zero. The converse intuition applies at the upper end of the payoff distribution.

Selection of the Numeraire Good An important practical consideration for the EEM(m) is the selection of the numeraire commodity. Different choices of the numeraire good lead to different versions of the mechanism. As shown by Proposition 5 below, in general at most one good can serve as the numeraire. That is, either the EEM(m) is not feasible for every candidate numeraire or the numeraire good is uniquely determined.

Definition 1. Consider the allocation function (5a). An allocation problem *admits extreme assignments* (AEA) if for each $m \in \{1, \ldots, n\}$, there exist announcements $\hat{v}_A$ and $\hat{v}_B$ such

[20] We have experimented with a large number of numerical examples similar to Examples 1 and 2. No case has refuted the conjecture that the EEM(m) is always superior to both ASM and DCM whenever it is feasible.

[21] As agents are risk neutral, only expected utilities matter to them. Nevertheless, a mechanism designer, whom we do not model directly, may entertain additional distributional concerns. For example, an "equitable" ex-post division is a common objective in a divorce proceeding.

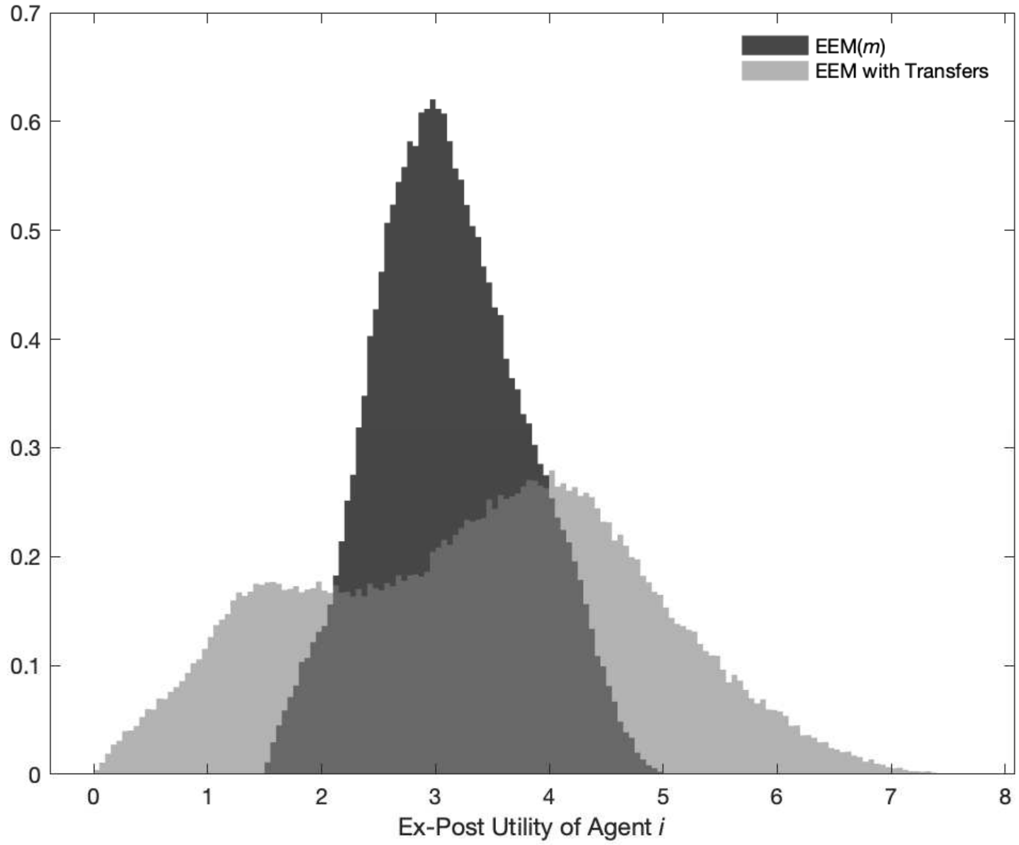

Figure 1: Estimates of the distribution of a representative agent's ex-post utility under the EEM(m) and the EEM with transfers in Example 2. Histograms are based on 100,000 simulated markets and are rescaled to unit area.

that $\mathbb{E}_{\tilde{v}_B}[q_{Bk}(\hat{v}_A, \tilde{v}_B)] = s_k$ for all $k \neq m$ and $\mathbb{E}_{\tilde{v}_A}[q_{Ak}(\tilde{v}_A, \hat{v}_B)] = 0$ for all $k \neq m$.

When an allocation problem satisfies the AEA property, an agent's announcement can (almost surely) guarantee that he receives all or no units of each good $k \neq m$ given that the other agent truthfully reveals his type. The allocation problem in Example 2 satisfies the AEA property.[22] More generally, the AEA property holds when agents are ex-ante symmetric and valuations are continuously distributed, an important benchmark situation.

Proposition 5. *Consider an allocation problem satisfying AEA. If the EEM(m) is feasible, then there will be precisely one good that can serve as the numeraire.*

Proof. We argue by contradiction. Suppose that the mechanism is feasible using either good m or good m' as the chosen numeraire. Since the EEM(m) is feasible, $0 \leq q_{Am}(\hat{v}_A, \hat{v}_B) \leq s_m$ for all $(\hat{v}_A, \hat{v}_B)$. This implies

$$\mathbb{E}_{\tilde{v}_B}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{Bk}}{\tilde{v}_{Bm}}\right) q_{Bk}(\hat{v}_A, \tilde{v}_B)\right] - \mathbb{E}_{\tilde{v}_A}\left[\sum_{k \neq m}\left(\frac{\tilde{v}_{Ak}}{\tilde{v}_{Am}}\right) q_{Ak}(\tilde{v}_A, \hat{v}_B)\right] \leq \frac{s_m}{2}.$$

Since the allocation problem admits extreme assignments,

$$\mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm'}}{\tilde{v}_{Bm}}\right] s_{m'} \leq \frac{s_m}{2}. \tag{7}$$

And likewise when m' assumes the role of numeraire,

$$\mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm}}{\tilde{v}_{Bm'}}\right] s_m \leq \frac{s_{m'}}{2}. \tag{8}$$

Combining (7) and (8),

$$2\mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm'}}{\tilde{v}_{Bm}}\right] \mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm}}{\tilde{v}_{Bm'}}\right] s_m \leq \frac{s_m}{2},$$

[22] To check whether the AEA property applies, it is often sufficient to examine extreme valuation reports. Consider Example 2 and suppose $m = 1$. If $\hat{v}_A = (2, 1)$ and $\hat{v}_B = (1, 2)$, then $\mathbb{E}_{\tilde{v}_B}[q_{B2}(\hat{v}_A, \tilde{v}_B)] = s_2$ and $\mathbb{E}_{\tilde{v}_A}[q_{A2}(\tilde{v}_A, \hat{v}_B)] = 0$. When $m = 2$, $\hat{v}_A = (1, 2)$ and $\hat{v}_B = (2, 1)$ satisfy the criterion.

which implies

$$\mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm'}}{\tilde{v}_{Bm}}\right]\mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{v}_{Bm}}{\tilde{v}_{Bm'}}\right] \leq \frac{1}{4}.$$

The final expression is a contradiction since the left hand side of the inequality exceeds 1. (Recall that for random variable X, $\mathbb{E}[X]\mathbb{E}[1/X] \geq 1$.) □

6 Generalizations

In this section we briefly describe two generalizations of our model in the absence of money.

6.1 More than Two Agents

For expositional ease, the analysis above has assumed two agents. The EEM and EEM(m) naturally and easily generalizes to the case where $N > 2$ agents get to divide up the available goods. Focusing on the case without monetary transfers, the allocation functions (5a) and (5b) generalize as follows. Fix a numeraire commodity m and given the profile of the agents' announced valuations, $\hat{v} := (\hat{v}_A, \hat{v}_B, \ldots)$, let $M_k(\hat{v})$ be the set of agents announcing the highest valuation ratio $\hat{v}_{ik}/\hat{v}_{im}$ for good k. Allocate

$$q_{ik}(\hat{v}) = \begin{cases} \frac{s_k}{|M_k(\hat{v})|} & \text{if } i \in M_k(\hat{v}) \\ 0 & \text{otherwise} \end{cases}$$

units of each good $k \neq m$ to agent i. For good m, let

$$q_{im}(\hat{v}) = \frac{s_m}{N} + \sum_{j\neq i}\mathbb{E}_{\tilde{v}_{-i}}\left[\sum_{k\neq m}\left(\frac{\tilde{v}_{jk}}{\tilde{v}_{jm}}\right)q_{jk}(\hat{v}_i, \tilde{v}_{-i})\right] - \frac{1}{N-1}\sum_{\ell\neq i}\mathbb{E}_{\tilde{v}_{-\ell}}\left[\sum_{j\neq\ell}\sum_{k\neq m}\left(\frac{\tilde{v}_{jk}}{\tilde{v}_{jm}}\right)q_{jk}(\hat{v}_\ell, \tilde{v}_{-\ell},)\right].$$

The preceding expression has three parts. As in (5b), we initially "endow" agent i with s_m/N units of good m. The second term represents the expected externality expressed in valuation ratios. The final term ensures budget balance. It directly adapts the analogous construction found in d'Aspremont and Gérard-Varet (1979).

When there is sufficient quantity of good m to implement the required transfers, Propositions 3 and 5 continue to apply. The proofs are essentially identical to the two-person case and we omit them for brevity.

6.2 Nonlinear Preferences

The analysis above has assumed linear preferences for each good. This assumption is common in the prior literature on object assignment and division.[23] An important advantage of the EEM(m) is that it naturally generalizes to allow for nonlinear preferences as well as complementarity and/or substitutability among goods. In the dissolution of a multi-asset real estate partnership, for example, an individual receiving building 3 may have an enhanced preference for adjacent building 4. In the division of an estate containing two diamond bracelets, an heir may greatly value either one, but have little use for a second bracelet (Pratt and Zeckhauser, 1990).

Consider again a setting with two agents, but posit that the utility of agent i given assignment $q_i = (q_{i1}, \ldots, q_{in})$ is

$$u_i(q_i|v_i) = \nu_i(q_{i1}, \ldots, q_{im-1}, q_{im+1}, \ldots q_{in}) + v_{im}q_{im}.$$

In the preceding expression, the agent's (private) type is $v_i = (\nu_i, v_{im})$; it consists of a function $\nu_i \colon \mathbb{R}^{n-1} \to \mathbb{R}$ and a positive scalar v_{im}. The function $\nu_i(\cdot)$ takes the vector

$$q_{(i,-m)} := (q_{i1}, \ldots, q_{im-1}, q_{im+1}, \ldots q_{in})$$

[23] For example, see Hylland and Zeckhauser (1979), McAfee (1992), or Miralles (2012).

as its argument. This function captures various forms of complementarity and substitutability among all goods other than m. Good m, which will serve as a numeraire commodity, enters an agent's valuation function linearly with a private marginal utility of v_{im}.

The allocation functions (5a) and (5b) generalize naturally to the class of preferences described above. Let $q_{(i,-m)}(\hat{v}_A, \hat{v}_B)$ be the allocation of all goods apart from good m to agent i given the announced types $\hat{v}_A = (\hat{\nu}_A, \hat{v}_{Am})$ and $\hat{v}_B = (\hat{\nu}_B, \hat{v}_{Bm})$. In this case, $q_{(A,-m)}(\hat{v}_A, \hat{v}_B)$ and $q_{(B,-m)}(\hat{v}_A, \hat{v}_B)$ together solve

$$\max_{q_{(A,-m)}, q_{(B,-m)}} \frac{\hat{\nu}_A(q_{(A,-m)})}{\hat{v}_{Am}} + \frac{\hat{\nu}_B(q_{(B,-m)})}{\hat{v}_{Bm}}$$

subject to the usual constraints: $0 \leq q_{Ak} + q_{Bk} \leq s_k \; \forall k \neq m$ and $q_{ik} \geq 0$ for each $i \in \{A, B\}$ and $k \neq m$. The resulting allocation of good m is

$$q_{Am}(\hat{v}_A, \hat{v}_B) = \frac{s_m}{2} + \mathbb{E}_{\tilde{v}_B}\left[\frac{\tilde{\nu}_B\left(q_{(B,-m)}(\hat{v}_A, \tilde{v}_B)\right)}{\tilde{v}_{Bm}}\right] - \mathbb{E}_{\tilde{v}_A}\left[\frac{\tilde{\nu}_A\left(q_{(A,-m)}(\tilde{v}_A, \hat{v}_B)\right)}{\tilde{v}_{Am}}\right].$$

The function $q_{Bm}(\hat{v}_A, \hat{v}_B)$ is defined analogously. It is straightforward to see that Lemma 1 and Proposition 3 continue to apply given the generalized allocation functions. We omit the proofs.

When preferences are nonlinear, the EEM(m) will often require that many goods apart from the numeraire be split among the agents. Following classic reasoning, when good $k \neq m$ is split between the agents, an efficient allocation must equalize the associated marginal utilities. In this case, the implemented allocation satisfies

$$\frac{\partial \nu_A\left(q_{(A,-m)}\right)}{\partial q_{Ak}} \Bigg/ \frac{\partial \nu_B\left(q_{(B,-m)}\right)}{\partial q_{Bk}} = \frac{v_{Am}}{v_{Bm}}.$$

As in section 5, the ratio v_{Am}/v_{Bm} plays an important role in pinning down the final assignment.[24]

The ability to implement allocations that equalize marginal utilities across many goods makes the EEM(m) well-suited for division problems with complex, nonlinear preferences. Alternative schemes can easily stumble in these cases. For instance, the ASM allocates all units of a good to one agent. Except in special circumstances, such an "all-or-nothing" assignment is unlikely to be ex-post efficient. Equal division almost surely fails on this dimension too.

7 Conclusion

We study the problem of allocating n goods to two agents whose cardinal preferences are private information. While the agents' types are independent, an agent may have correlated values for the goods. If money is available as a medium of exchange, Bayesian incentive compatibility and ex-ante Pareto efficiency can be achieved, thus implying ex-post Pareto efficiency.

If money is not available as a medium of exchange, Bayesian incentive compatibility and ex-post Pareto efficiency remain achievable using a modification of the expected externality mechanism under certain reasonable conditions. That mechanism, which we call the EEM(m), uses one of the goods as a numeraire good instead of money. Each agent is allocated the goods that he values more than the other agent relative to the designated numeraire good. As a result, no mutually beneficial trades are available after allocations are made and ex-post Pareto efficiency is achieved.

[24] Occasionally the EEM(m) outcome may necessitate allocating all units of a particular good to one agent, as in the linear case. When agent A receives all of good j,

$$\frac{\partial \nu_A\left(q_{(A,-m)}\right)}{\partial q_{Ak}} \bigg/ \frac{\partial \nu_B\left(q_{(B,-m)}\right)}{\partial q_{Bk}} > \frac{v_{Am}}{v_{Bm}}\ .$$

The inequality is flipped when agent B receives all of good k. The ratio v_{Am}/v_{Bm} again defines the threshold "θ" for the goods' ex-post efficient assignment. See the discussion in section 5 and footnote 14.

Across a broad array of situations, individuals must divide up a collection of goods in a fair and efficient manner. The classic case in the literature is dividing up an estate. In everyday life, this situation arises frequently when those who hold assets collectively, as in a partnership or marriage, wish to convert to individual ownership of assets, e.g., to dissolve a partnership or terminate the marriage. Hence, we think that attention to the expected externality mechanism as a means to pursue the goals of fairness and efficiency should have broad application.

The elicitation of preferences and the calculation of allocations for the EEM and the EEM(m) can get complex. That suggests that in actual operation they may be computer mediated. Interestingly, expected externality methods may hold promise for a range of Internet-related applications.[25] The mechanism design approach has been widely employed in the Internet-mediated world, for example in the conduct of auctions for advertising. That same world, given a modest dose of technological advance, would seem to hold the potential to vastly expand the real world applications of the fair division approach. If that potential is realized, there should be substantial spillovers to an array of legal settings, such as divorces and partnership dissolutions.

[25]A telling anecdote has been provided to us by Chris Robert, CEO of Dobility, a small Internet-focused company whose major product, SurveyCTO, enables the conduct and analysis of surveys. His company gives clients a specified quantity of free survey submissions, and then imposes a constant cost per additional submission. The goal is to keep "overall demand for downstream server resources (CPU, memory, and disk resources) manageable.... We try to put guardrails around usage...I realize that a much better system would follow the fair division approach. It would elicit from people a set of preferences that they themselves control, which would then define various personal limits and guardrails." (Personal communication, February 16, 2019). The scale of operations on the Internet may also allow the mediator or mechanism designer to estimate the valuation distributions, which are an important input to the mechanisms' formulation.

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