First-Price Auctions with Budget Constraints

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Abstract

Consider a first-price, sealed-bid auction with interdependent valuations and private budget constraints. Private budget constraints introduce subtle strategic tradeoffs with first-order consequences for equilibrium bidding. In a pure-strategy, symmetric equilibrium, agents may adopt discontinuous bidding strategies resulting in a stratification of competition along the budget dimension. In an asymmetric setting, equilibria in "nondecreasing" strategies exist, albeit in a qualified sense. Private budgets introduce significant confounds for the interpretation of bidding data due to their interaction with risk preferences and their countervailing strategic implications.

Keywords: First-Price Auctions, Budget Constraints, Interdependent Values, Asymmetric Auctions, Monotone Equilibrium
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1 Introduction

Private budgets arise in many economic situations, including auctions. Once bidders have multidimensional private information involving a value-signal, which informs their preferences, and a budget, which defines their feasible strategy set, even simple bidding games exhibit nontrivial equilibria. Many familiar intuitions require qualification or reassessment.

For concreteness, consider a first-price, sealed-bid auction for one item. Bidders simultaneously place bids, the highest bidder wins, and he pays his own bid. Private budgets affect equilibrium bidding in two opposing ways in this familiar setting. First, budgets dampen bids by introducing a spending limit. Che and Gale (1998) characterize this feature in their seminal analysis of the first-price auction with private budget constraints.[1] Assuming private values, they propose sufficient conditions ensuring equilibrium bidding strategies assume the following *canonical form*:

$$\beta(s_i, w_i) = \min\left\{\bar{b}(s_i), w_i\right\}. \tag{1}$$

In (1), s_i is bidder i's private value, w_i is his private budget, and $\bar{b}(\cdot)$ is an increasing, continuous function that equals the bid of an "unconstrained" agent who bids less than his budget.

Second, and less intuitively, budgets can amplify certain agents' equilibrium bids. A higher bid may be disproportionally effective at defeating budget-constrained rivals, thus giving a high-budget bidder a new strategic option. Exercising this option can lead to an equilibrium where agents adopt discontinuous bidding strategies, even in a symmetric setting. A stratification of competition between high- and low-budget bidders emerges in equilibrium. Hitherto under-appreciated in the literature, these strategic implications complicate equilibrium bidding and affect the auction's efficiency and revenue potential.

To illustrate our argument, it is helpful to consider an example conveying its central intuition. The following is an extension of Vickrey's (1961) model.[2]

Example 1. Consider a first-price, sealed-bid auction for one item. There are two ex ante identical bidders. Each bidder i observes an independent, uniformly-distributed signal $s_i \in [0, 1]$ regarding his value for the item. Signals are private, but their distribution is common knowledge. If a bidder with value-signal s_i wins the auction with the bid b_i, his payoff is $s_i - b_i$; otherwise, it is zero. A fair coin flip resolves ties. It is well-known that in this auction's Bayesian-Nash equilibrium both bidders adopt the strategy $s_i/2$.

[1] Most related to our study is the model found in Section 3 of Che and Gale (1998).
[2] Milgrom (2004) presents a related example with a common budget constraint.

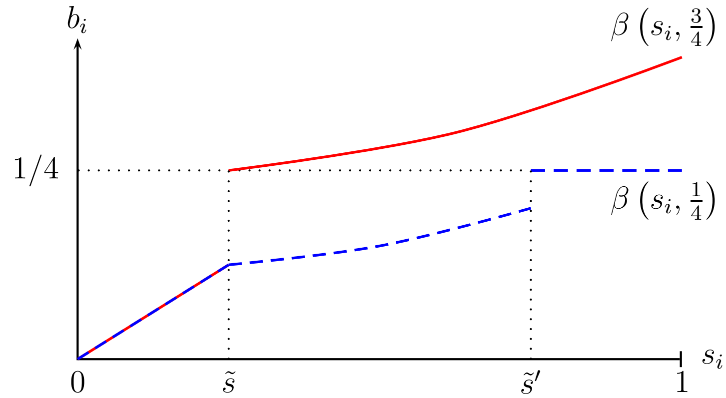


Figure 1: The equilibrium strategy in Example 1. Figure not to scale.

Suppose, additionally, that each bidder has a budget w_i, which is independent of his value-signal. Budgets are private, but their distribution is common knowledge. With probability 1/2, a bidder's budget is 1/4; with probability 1/2, it is 3/4. A budget is a hard constraint on bids and a type-(s_i, w_i) bidder cannot bid above w_i.

A tempting conjecture is that

$$\min\{s_i/2, w_i\} \tag{2}$$

is a symmetric equilibrium strategy in this enriched setting. In (2), which has the canonical form, a bidder follows the usual equilibrium strategy $s_i/2$ until he exhausts his funds. Though intuitive, (2) cannot be a symmetric equilibrium strategy. If bidder j follows (2), then bidder i has a profitable deviation when $s_i = 1/2 - \epsilon$ and $w_i = 3/4$. According to (2), he should bid $1/4 - \epsilon/2$. But, if he bids ϵ more, $1/4 + \epsilon/2$, his probability of winning jumps from approximately 1/2 to over 3/4. At infinitesimal added cost, this deviation is worth it.[3]

Reflecting on the preceding reasoning, we are led to an equilibrium strategy with jump discontinuities. Figure 1 sketches this auction's symmetric equilibrium strategy,

$$\beta(s_i, 1/4) = \begin{cases} s_i/2 & \text{if } s_i \in [0, \tilde{s}] \\ \frac{1+9s_i^2}{6+18s_i} & \text{if } s_i \in (\tilde{s}, \tilde{s}'] \\ 1/4 & \text{if } s_i \in (\tilde{s}', 1] \end{cases} \qquad \beta(s_i, 3/4) = \begin{cases} s_i/2 & \text{if } s_i \in [0, \tilde{s}] \\ \frac{5+9s_i^2}{18+18s_i} & \text{if } s_i \in (\tilde{s}, 1] \end{cases}$$

where $\tilde{s} = 1/3$ and $\tilde{s}' = 11/27$.[4] Clearly, this strategy departs from the canonical form.

[3] Similar reasoning applies whenever the bid distribution has a mass point due to a mass point in the budget distribution in the relevant range of bids. When this is the case, the first-price auction cannot have an equilibrium in continuous strategies.

[4] We verify that the defined strategy is a symmetric equilibrium in Online Appendix A. We show that the equilibrium's qualitative characteristics are unchanged when the distribution of valuations is $H(s)$ and

Both dampening and strategic consequences of private budgets are apparent in this example's equilibrium. The former is obvious. A high-valuation ($s_i > \tilde{s}'$), low-budget ($w_i = 1/4$) bidder hits his spending limit in equilibrium—his bid is capped.

More interesting and consequential are the strategic implications. First, a high-budget bidder's strategy features a prominent jump discontinuity that amplifies his bid at $\tilde{s}$. A high-budget bidder has the option of bidding above 1/4, thus outbidding with certainty a low-budget competitor. Exercising this option becomes worthwhile once his value exceeds $\tilde{s}$. The intuition for the discontinuity at $\tilde{s}'$ is similar. Given that low-budget agents cannot bid above 1/4, there is a chance of a tie at this bid. A low-budget bidder with valuation $s_i > \tilde{s}'$ finds this outcome sufficiently attractive to actually bid 1/4. He would like to bid more to break the (possible) tie in his favor, but lacks the funds to do so.

Second, despite occasional jumps, on the margin bidding is less brazen due to a stratification of competition along the budget dimension. When an agent bids above 1/4, he need only worry about a competing bid from another high-budget bidder, which occurs with probability less than 1/2. Similarly, in equilibrium, a low-budget bidder of type $s_i > \tilde{s}$ competes on the margin only against a low-budget adversary.[5] In each case, there is less incentive to bid aggressively since segregation has reduced competition in the relevant range of bids.

Although the preceding observations appear to be specific to Example 1, or are perhaps an artifact of the discrete budget distribution, this is not the case. As explained in Section 3, the strategic amplification and the competitive stratification can arise even when budgets assume a continuum of values. Necessarily the construction is more elaborate, but the intuition is the same. Focusing on the two-bidder case, we propose new sufficient conditions for the existence of an equilibrium satisfying the canonical form (1), even when values are interdependent. Relaxing these assumptions leads to a non-canonical equilibrium echoing Example 1's intuition. The resulting empirical confounds can be substantial. For example, a uniform distribution of value-signals and budgets may result in a bimodal equilibrium bid distribution (Example 2).

Whereas our baseline two-bidder model isolates the strategic implications of private budgets, it does not span all relevant confounds introduced by this variable. In Section 4 we consider an N-bidder, non-quasilinear, asymmetric setting. Monotone equilibria in nonde-

budgets assume low ($\underline{w}$) and high ($\overline{w}$) values with probabilities p and $1 - p$, respectively.

[5] Consider the equilibrium bid of agent i with value-signal $s_i \in (\tilde{s}, \tilde{s}')$. When he bids $\beta(s_i, 1/4)$, bidder i wins the auction when $s_j < s_i$ and $w_j = 1/4$ or $s_j < \tilde{s}$ and $w_j = 3/4$. If bidder i bids slightly more, he now wins when $s_j < s_i + \epsilon$ and $w_j = 1/4$ or $s_j < \tilde{s}$ and $w_j = 3/4$. Agent i's higher bid is effective only against a low-budget adversary.

creasing strategies, as traditionally defined, may not exist, due to countervailing incentives associated with the budget constraint variable. Increasing a bidder's budget may imply a higher or a lower equilibrium bid. Applying methods introduced by Reny (2011) for the analysis of games with multidimensional private information, we show that an equilibrium exists in "nondecreasing" strategies when these are *re*defined to account for the interaction between budgets and preferences. Monotone equilibria are focal in an auction and provide a minimal benchmark for understanding its economic properties.

Our study has three main contributions. First, our results offer guidance concerning bidding in first-price auctions by isolating the strategic implications of private budgets. The importance of the strategic dimension is surprising given the simplicity of the first-price sealed-bid auction format.

Second, we offer a cautionary message for empirical studies. The conflicting dampening and amplifying responses to the budget constraint variable imply that bids need not be skewed in one direction. Local incentive constraints, which underly many empirical methods for auction analysis, need not fully characterize equilibrium bidding. Even small changes in a bidder's type may lead to large changes in his equilibrium bid. Multiple equilibria are possible too (Example 3). We hope our analysis provides a useful step toward better integrating budgets in empirical analysis of auctions.

Third, our methods may be useful in further investigations of auctions with budget constraints or with other forms of multidimensional private information. The phase-plane analysis employed in Section 3 is both simple and informative of associated economic incentives. Likewise, the type-space ordering used in Section 4 may aid in the analysis of other auctions or contests with multidimensional private information.

Outline Section 2 surveys the related literature. Section 3 generalizes Example 1 to allow for interdependent valuations and continuously distributed budgets. We examine both canonical equilibria and non-canonical, discontinuous equilibria. Section 4 extends our model to the asymmetric case. Omitted proofs are collected in Appendices A, B, and C. An Online Appendix contains examples and technical extensions omitted for brevity.

2 Related Literature

Che and Gale (1998) were the first to study standard auctions with private values and private budget constraints.[6] They investigate equilibria satisfying the canonical form (1). In Section 3.1 we study canonical equilibria in a complementary class of cases, including those with interdependent values. Fang and Parreiras (2002, 2003) study the second-price auction with private budgets. We discuss their results in detail in Section 3.4. In a similar setting, Kotowski and Li (2014a,b) analyze all-pay auctions. Ghosh et al. (2018) study wars of attrition with private budgets.

In developing our model, we suppress several embellishments commonly associated with budget constraints. Unlike Zheng (2001), Jaramillo (2004), or Rhodes-Kropf and Viswanathan (2005), we model budgets as hard bounds on bids, rather than as imperfections in bid-financing ability. We also abstract from the risk of collusion or the formation of bidding consortia due to liquidity limitations (Cho et al., 2002). Burkett (2016) argues that budget constraints solve a principal-agent problem when bidding is delegated to a third party. Budgets are exogenous in our model, as they would be in the absence of delegation.

Interest in the role of budgets in auctions has grown due to their appearance in many Internet-related applications (Ashlagi et al., 2010; Balseiro et al., 2015) and spectrum auctions (Cramton, 1995; Bulow et al., 2017). Noting these applications, several authors build models investigating budget constraints in sequential or multi-unit auctions (Pitchik and Schotter, 1988; Benoît and Krishna, 2001; Pitchik, 2009; Brusco and Lopomo, 2008, 2009; Ghosh, 2015; Kariv et al., 2018). Our model is not a special case of any of them.

The auction design problem following Myerson (1981) incorporating budget constraints is particularly challenging and has been considered under various guises (Laffont and Robert, 1996; Monteiro and Page, 1998; Che and Gale, 1999, 2000; Maskin, 2000; Malakhov and Vohra, 2008; Pai and Vohra, 2014; Kojima, 2014; Baisa, 2018; Boulatov and Severinov, 2018; Carbajal and Mu'alem, 2018; Richter, 2019). Auction design with budgets has also spurred an interest among computer scientists.[7] Our analysis takes the auction format as given and examines equilibrium bidding. Che and Gale (2006) examine auction revenue when bidders have multidimensional types, which may include private financial constraints.

[6] Che and Gale (1996a,b) are precursors that examine a common-value setting where only budgets are private information. Bobkova (2019) generalizes these models to an asymmetric setting. Kariv et al. (2018) corroborate the equilibrium predictions of Che and Gale (1996a) with a laboratory experiment.

[7] Borgs et al. (2005) and Dobzinski et al. (2012) are representative contributions.

3 Symmetric Auctions with Interdependent Values

Consider the following adaptation of Milgrom and Weber's (1982) model of a first-price auction. There are two ex ante symmetric bidders. Let $s_i \in [0,1]$ be the value-signal of bidder i. It defines his information about the item for sale. Value-signals are independently and identically distributed according to the twice continuously-differentiable cumulative distribution function (c.d.f.) $H(s_i)$. (We relax independence in Remark 3 below.) The associated probability density function (p.d.f.) $h(s_i)$ is strictly positive and bounded. Given s_i and s_j, $v(s_i, s_j)$ is bidder i's valuation for the item. The valuation function is continuously differentiable, strictly increasing in the first argument, nondecreasing in the second, $v(0,0) = 0$, and $v(1,1) = \overline{v}$. If bidder i wins the auction with the bid b_i, his payoff is $v(s_i, s_j) - b_i$; otherwise, it is 0. Ties are resolved with a uniform randomization.

When budget constraints are absent, Milgrom and Weber (1982) show that there exists a Bayesian-Nash equilibrium where all bidders adopt a common strategy, $\alpha(s_i)$, that solves

$$\alpha'(s_i) = (v(s_i, s_i) - \alpha(s_i))\frac{h(s_i)}{H(s_i)} \tag{3}$$

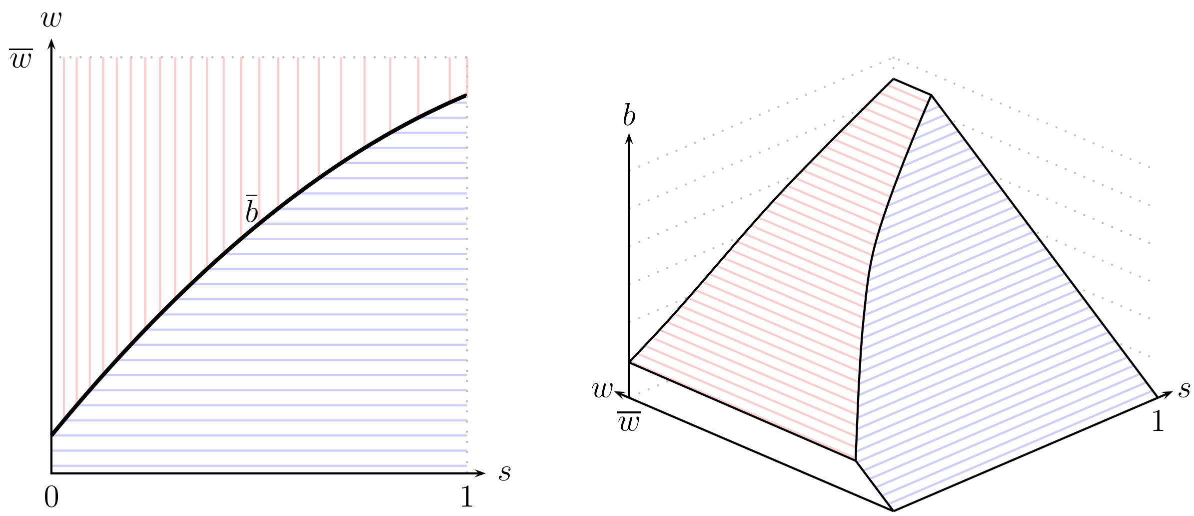
subject to $\alpha(0) = 0$. Henceforth, we reserve the notation "$\alpha(\cdot)$" for this specific strategy.

Generalizing the model, let $w_i \in [\underline{w}, \overline{w}]$ be the budget of bidder i. Budgets are independently and identically distributed according to the twice continuously-differentiable c.d.f. $G(w_i)$. We assume that $0 \leq \underline{w} < \overline{v} < \overline{w}$ and that the associated p.d.f. $g(w_i)$ is strictly positive and bounded for all $w_i \in [\underline{w}, \overline{w}]$. For convenience, we define $G(w_i) = 0$ if $w_i < \underline{w}$ and $G(w_i) = 1$ if $w_i > \overline{w}$. A bidder's type (s_i, w_i) is private information; otherwise, the environment is common knowledge. In this section we focus on a symmetric Bayesian-Nash equilibrium and we suppress bidder subscripts.

3.1 A Canonical Equilibrium

Positing private values, Che and Gale (1998) identify a sufficient condition (see Remark 2 below) ensuring that the first-price auction has an equilibrium where bidders adopt a common canonical strategy, $\beta(s, w) = \min\{\bar{b}(s), w\}$. The function $\bar{b}(s)$, which is strictly increasing and continuous, equals the bid of an "unconstrained" or "high-budget" bidder who bids less than his budget in equilibrium. Figure 2 illustrates a representative canonical strategy.

We first investigate conditions ensuring a canonical equilibrium exists in our model allowing for interdependent values. As an initial step, we derive some properties of $\bar{b}(\cdot)$. If



(a) Level sets of $\beta(s, w)$.

(b) Three dimensional sketch of $\beta(s, w)$.

Figure 2: Illustration of a canonical strategy, $\beta(s, w) = \min\{\bar{b}(s), w\}$, when $\underline{w} = 0$.

bidder j follows a canonical strategy, the expected payoff of a type-(s, w) bidder i when he bids $\bar{b}(x) \leq w$ is

$$\begin{aligned} U(\bar{b}(x)|s, w) = &\int_0^x \int_0^{\overline{w}} \big(v(s, y) - \bar{b}(x)\big)\, h(y)g(z)\ dzdy \\ &+ \int_x^1 \int_0^{\bar{b}(x)} \big(v(s, y) - \bar{b}(x)\big)\, h(y)g(z)\ dzdy. \end{aligned} \tag{4}$$

The first term in (4) is the expected payoff from defeating an opponent with a value-signal less than x. The second term is the expected payoff from defeating an opponent with a value-signal greater than x who has a budget less than $\bar{b}(x)$. This term vanishes if $\bar{b}(x) < \underline{w}$. By computing the first-order condition, $dU(\bar{b}(x)|s, w)/dx\big|_{x=s} = 0$, and defining the expressions

$$\begin{aligned} \lambda(s) &:= \frac{h(s)}{1 - H(s)}, \\ \gamma(b) &:= \frac{g(b)}{1 - G(b)}, \\ \eta(x, s) &:= \int_x^1 \frac{v(s, y)h(y)}{1 - H(x)} dy, \qquad \text{and} \\ \delta(b, s) &:= b + \frac{G(b)}{g(b)} + \frac{H(s)}{g(b)(1 - H(s))}, \end{aligned}$$

we can characterize $\bar{b}(s)$ at its points of differentiability as follows:

$$\bar{b}'(s) = \begin{cases} \left(v(s,s) - \bar{b}(s)\right) \dfrac{h(s)}{H(s)} & \text{if } \bar{b}(s) < \underline{w} \\ \dfrac{\lambda(s)}{\gamma(\bar{b}(s))} \left(\dfrac{\bar{b}(s) - v(s,s)}{\eta(s,s) - \delta(\bar{b}(s), s)} \right) & \text{if } \bar{b}(s) > \underline{w} \end{cases}. \tag{5}$$

When $\bar{b}(s) < \underline{w}$, (5) reduces to (3), the differential equation characterizing the equilibrium strategy in the absence of budget constraints. To interpret (5) when $\bar{b}(s) > \underline{w}$, we expand the equation and rearrange its terms. Omitting the function arguments for readability,

$$\underbrace{h(1-G)(v-\bar{b})}_{[A]} + \underbrace{\bar{b}'g(1-H)(\eta - \bar{b})}_{[B]} = \underbrace{\bar{b}'(G(1-H)+H)}_{[C]}. \tag{6a}$$

Terms [A] and [B] are the expected marginal benefit of an infinitesimally higher bid. When the higher bid defeats an opponent with a value-signal less than s, the payoff gain is $v(s,s) - \bar{b}(s)$; otherwise, when it defeats an opponent with a budget less than $\bar{b}(s)$, the payoff gain is $\eta(s,s) - \bar{b}(s)$.[8] Term [C] is the expected marginal cost of a higher bid.

Further rearrangement of (6a) gives

$$\bar{b}' = \frac{h(1-G)(v-\bar{b})}{(G(1-H)+H) - g(1-H)(\eta - \bar{b})}. \tag{6b}$$

For $\bar{b}'(s)$ to be positive, as assumed throughout its derivation, both numerator and denominator in (6b) must have the same sign. When the budget distribution is relatively dispersed, and say $g(b) \approx 0$, this is not a demanding requirement as the denominator is positive. If the budget distribution is concentrated near some value, and $g(b)$ is relatively large, the denominator of (6b) may approach zero, or be negative. Intuitively, when this occurs the likelihood of defeating a budget-constrained opponent is large and a (high-budget) bidder wishes to increase his bid aggressively to capitalize on this opportunity. In Example 1, the return to a slightly-higher bid spiked at the atom in the budget distribution at $1/4$ and lead to a discontinuous jump in a bid. As (6b) reveals, an atom is not necessary to provoke a similar response.

Motivated by the preceding discussion, we introduce a regularity condition on the denominator of (5) ensuring the marginal returns to a slightly higher bid are moderate and

[8] The term $\eta(x,s)$ is the expected value of the item to a bidder with value-signal s conditional on defeating an opponent with a value-signal greater than x.

well-behaved. As explained below, it is a sufficient condition for a canonical equilibrium to exist when $\underline{w} = 0$.

Assumption 1. *For each s, the function $b \mapsto \eta(s,s) - \delta(b,s)$ crosses zero at most one time. Furthermore, if $\eta(s',s') - \delta(b',s') = 0$, then $\partial\delta(b,s')/\partial b|_{b=b'} > 0$.*

Remark 1. Assumption 1 holds when the budget c.d.f. $G(\cdot)$ is concave.

Remark 2. Assumption 1 is implied by Assumption 5 of Che and Gale (1998) when the latter is applied to the case of independent budgets and value-signals, as in our model.[9]

Assumption 1 has two parts. The first says that the denominator of (5) satisfies a single-crossing condition. Since $\eta(s,s) - \delta(\overline{w},s) < 0$, any crossing must be from above. Thus, the second part is a technical restriction ensuring the crossing is strict.

Theorem 1. *Suppose Assumption 1 holds and $\underline{w} = 0$. There exists an equilibrium where each bidder adopts a common strategy, $\beta(s,w) = \min\{\bar{b}(s), w\}$. The function $\bar{b}(\cdot)$ is increasing, continuous, and for a.e. s,*

$$\bar{b}'(s) = \frac{\lambda(s)}{\gamma(\bar{b}(s))}\left(\frac{\bar{b}(s) - v(s,s)}{\eta(s,s) - \delta(\bar{b}(s),s)}\right). \tag{7}$$

Generically, the equilibrium is unique in the class of symmetric equilibria in canonical strategies.

We provide a numerical example illustrating an equilibrium in Section 3.3. We devote this subsection's remainder to the proof of Theorem 1.

Construction of a Canonical Equilibrium To prove Theorem 1, we must confirm the function $\bar{b}(\cdot)$ exists with the stated properties. A natural approach is to (try to) solve (7) assuming $\bar{b}(0) = 0$. This is premature. With interdependent values, zero may not be an optimal bid for an unconstrained bidder. A small bid may defeat a budget-constrained opponent with favorable information concerning the item's value. This attenuation of the winner's curse can inflate an agent's bid, even if he observes a low value-signal. Moreover, (7) may lack an increasing solution that is defined for all s, as assumed throughout its derivation.

[9] Che and Gale (1998) invoke their Assumption 5 to prove their Lemma 1. Fang and Parreiras (2002, footnote 7) point out that Che and Gale's lemma may not be true without additional assumptions in their model. When applied to our model, Che and Gale's assumption is sufficient for the existence of an equilibrium in canonical strategies when $\underline{w} = 0$.

Acknowledging the above complications, we identify $\bar{b}(\cdot)$ indirectly by focusing on the qualitative behavior of solutions to (7). To do so, we use phase-plane analysis to study the closely-related plane-autonomous system

$$\dot{s}(s,b) = \gamma(b)(\eta(s,s) - \delta(b,s)) \qquad \dot{b}(s,b) = \lambda(s)(b - v(s,s)) \tag{8}$$

where $(s,b) \in [0,1] \times [\underline{w}, \overline{w}]$.[10] In analyzing this system, we are interested in the graphs of its solutions and not in its dynamics. Graphs of its solutions can be identified with integral curves of (7) since, following Birkhoff and Rota (1978, Chapter 5), we observe that

$$\frac{db}{ds} = \frac{db}{dt}\Big/\frac{ds}{dt} = \frac{\dot{b}(s,b)}{\dot{s}(s,b)} = \bar{b}'(s).$$

Interpreting the problem as proposed lets us use graphical techniques to discern the behavior of candidate solutions over a large domain (Strogatz, 1994; Perko, 2000). Additionally, our reinterpretation placates the singularities encountered in a direct analysis of (7).

Analysis of system (8) focuses on its nullclines and critical (or fixed) points. The *nullclines* are

$$\psi(s) := \{b \mid \eta(s,s) - \delta(b,s) = 0\} \quad \text{and} \quad \nu(s) := \{b \mid b - v(s,s) = 0\}.$$

These graphs characterize points where $\dot{s} = 0$ and $\dot{b} = 0$, respectively. Their economic interpretation is the following. At $b = \psi(s)$ the denominator of (7) changes sign. When $\eta(s,s) - \delta(b,s) > 0$, marginally higher bids are relatively effective (conditional on s) at defeating a budget-constrained opponent. Otherwise, this dimension's marginal contribution to a bidder's payoff is more subdued. By definition, $\nu(s) = v(s,s)$. Thus, $\nu(s)$ is a type-s bidder's valuation conditional on his opponent observing the same value-signal. If a type-s agent bids more than $\nu(s)$, then he is overbidding conditional on defeating only an opponent with a value-signal less than his own. A *critical point* occurs where nullclines intersect.

Assumption 1 ensures that the analysis of system (8) is routine. Figure 3 illustrates a representative case. Six facts are noteworthy.

(i) The curve $\nu(s)$ runs from the origin to the figure's top-right corner. In the region below $\nu(s)$, $\dot{b} < 0$; otherwise, $\dot{b} > 0$.

[10] The functions $\dot{s}$ and $\dot{b}$ are not defined at the boundaries where $b = \overline{w}$ and $s = 1$, respectively. This technical fact is not a concern and we suppress it for expositional clarity. We can perform the analysis on the domain's interior. The identified solution can be extended to the boundary using standard results in the theory of ordinary differential equations.

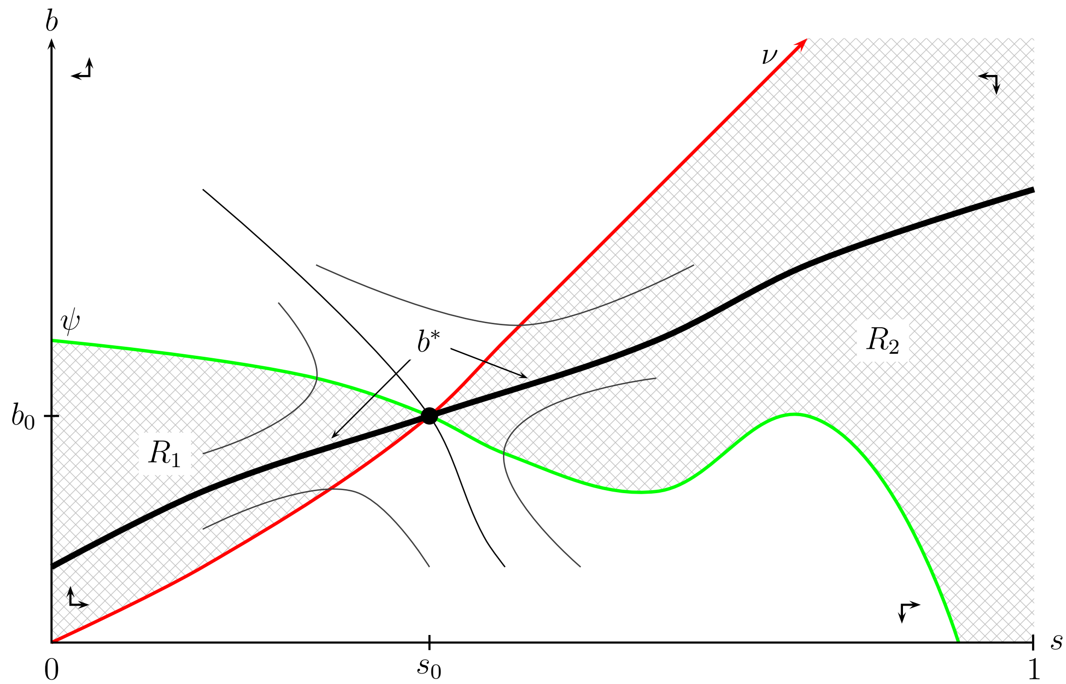


Figure 3: A phase-portrait of system (8) with one critical point.

(ii) Assumption 1 implies that $\psi(s)$ is at most single-valued. When not empty, $\psi(s)$ is a continuous curve.[11] In the region below $\psi(s)$, $\dot{s} > 0$; otherwise, $\dot{s} < 0$.

(iii) There exists at least one critical point when $\underline{w} = 0$ (Lemma A.2 in Appendix A). For simplicity, Figure 3 shows an instance with one critical point at (s_0, b_0). We address the case of multiple critical points in Online Appendix B. Multiple critical points do not affect Theorem 1, though an extended analysis is needed to identify $\bar{b}(\cdot)$.

(iv) The ratio $\dot{b}/\dot{s}$ is positive at point (s, b) only when b is "in between" $\nu(s)$ and $\psi(s)$. These are regions R_1 and R_2 in Figure 3. In these regions the solutions of (8) can be identified with increasing functions of s. Elsewhere, solutions have negative slope and, therefore, cannot constitute part of an admissible solution for $\bar{b}(s)$.

(v) Generically, each critical point is either a saddle point or a node (Lemma A.3). When there is one critical point, it is a saddle point.

(vi) When there is one critical point, of particular interest are the stable manifolds that approach it from below and above in regions R_1 and R_2. (The Hartman-Grobman

[11] Continuity follows from the implicit function theorem. Lemma A.1 in Appendix A records additional properties of $\psi(s)$, including the fact that $\psi(s) < \overline{v}$.

theorem implies that these manifolds are generically unique.) In Figure 3, these are the bold curves connecting (s_0, b_0) with the boundaries. Henceforth, we use $b^*(s)$ to denote the union of these solution paths' closures as a function of s. The function $b^*(s)$ is strictly increasing, continuous, and an integral curve of (7).

We are now ready to identify the function $\bar{b}(\cdot)$. This function must be strictly increasing, continuous, and an integral curve of (7). When there is one critical point, as in Figure 3, only the stable solutions of (8) approaching the critical point—the function $b^*(s)$—satisfy the stated requirements. And so, in this case, we define $\bar{b}(s) \equiv b^*(s)$ for all s.

The identification of $\bar{b}(s)$ illustrates the competing incentives facing a high-budget bidder. Consider again Figure 3. When $\bar{b}(s)$ is confined to region R_1, a type-s bidder would like to bid more to capitalize on his opponent's (possible) budget constraint. His incentive to do so is moderated by the fact that he is overbidding conditional on defeating an adversary with a value-signal less than his own, i.e. $\bar{b}(s) > v(s, s)$. In region R_2, these incentives flip and $v(s, s) > \bar{b}(s)$.

A standard argument lets us confirm that Theorem 1 characterizes an equilibrium.[12]

Proof of Theorem 1. Suppose bidder j adopts the strategy, $\beta(s, w) = \min\{\bar{b}(s), w\}$ where $\bar{b}(s)$ satisfies the conditions stated in Theorem 1. (The preceding discussion has established the existence of this function.) The expected utility of bidder i of type (s, w) who bids $\bar{b}(x)$ is given by (4). Differentiating (4),

$$\frac{\partial U(\bar{b}(x)|s, w)}{\partial x} = g(\bar{b}(x))(1 - H(x)) \left[\frac{\lambda(x)}{\gamma(\bar{b}(x))} \big(v(s, x) - \bar{b}(x) \big) + \bar{b}'(x) \big(\eta(x, s) - \delta(\bar{b}(x), x) \big) \right].$$

Suppose $x > s$. Clearly, $\eta(x, x) \geq \eta(x, s)$ and $v(x, x) \geq v(s, x)$. Thus, $\partial U(\bar{b}(x)|s, w)/\partial x \leq \partial U(\bar{b}(x)|x, w)/\partial x = 0$. Hence, bidder i can increase his payoff by bidding less than $\bar{b}(x)$. A parallel argument shows that when $x < s$, bidder i can increase his payoff by bidding more than $\bar{b}(x)$. Together, the two cases imply that bidder i has no incentive to deviate from $\beta(s, w)$ to another bid in the range of $\bar{b}(\cdot)$.[13]

Instead, suppose bidder i bids outside the range of $\bar{b}(\cdot)$. Any bid strictly exceeding $\bar{b}(1)$

[12] See Fang and Parreiras (2002) and Kotowski and Li (2014a) for similar proofs in the case of the second-price and all-pay auctions with budget constraints.

[13] We have shown that a type-s bidder can improve his payoff by increasing his bid whenever it is less than $\bar{b}(s)$. If $\beta(s, w) = w \leq \bar{b}(s)$, then he cannot increase it above his budget constraint.

is dominated by $\bar{b}(1)$. The bid $b \leq \bar{b}(0)$ yields an expected payoff of

$$U(b|s,w) = \int_0^1 \int_0^b (v(s,y) - b)\, h(y) g(z) dz dy = G(b)(\eta(0,s) - b). \tag{9}$$

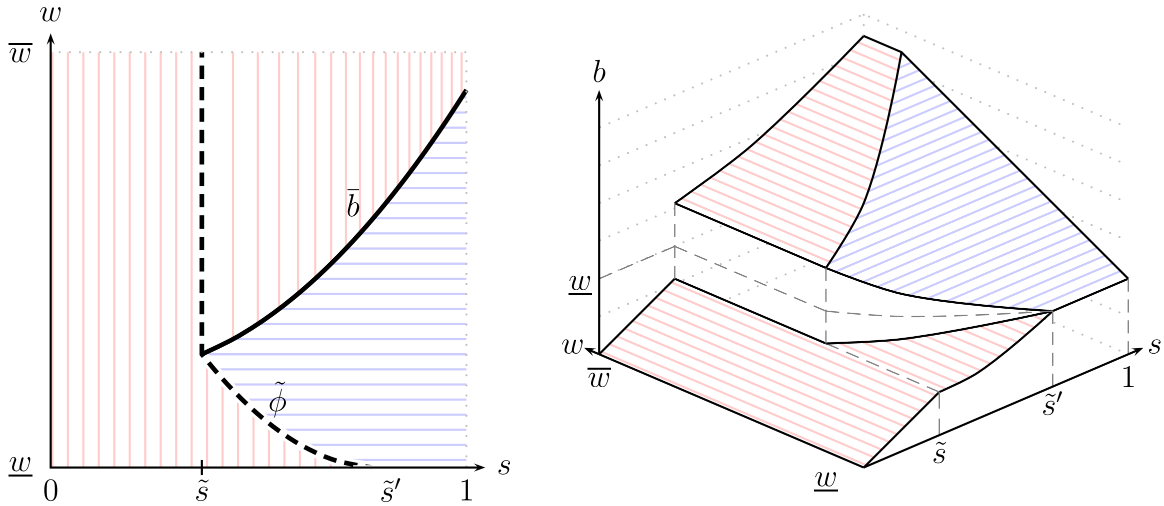
To complete the proof, it is sufficient to show that $\partial U(b|s,w)/\partial b \geq 0$ when $\underline{w} \leq b < \bar{b}(0)$. Differentiating (9), $\partial U(b|s,w)/\partial b = g(b)(\eta(0,s) - b) - G(b) \geq g(b)(\eta(0,0) - b) - G(b)$. Given the definition of $\bar{b}(\cdot)$, if $\bar{b}(0) > 0$ then $\psi(0) > \bar{b}(0)$. Recalling Assumption 1, $b < \psi(0)$ implies that $\eta(0,0) > \delta(b,0) = b + G(b)/g(b)$. Rearranging terms gives $g(b)(\eta(0,0) - b) - G(b) \geq 0$, which implies the desired conclusion. □

Remark 3. Theorem 1 generalizes to the case of affiliated value-signals (Milgrom and Weber, 1982). In this case, let $h(s_i, s_j)$ be the strictly positive, bounded, permutation-symmetric, and log-supermodular joint density of value-signals.[14] Define $h(x|s)$ and $H(x|s)$ as the conditional p.d.f. and c.d.f., respectively, and let $\lambda(x|s) := h(x|s)/(1 - H(x|s))$. The analogue of Theorem 1 holds if $v(\cdot, x)\lambda(x|\cdot)$ is nondecreasing, a common assumption (Krishna and Morgan, 1997; Lizzeri and Persico, 2000; Fang and Parreiras, 2002). This condition holds when, for example, $v(s_i, s_j) = (s_i + s_j)/2$ and $h(s_i, s_j) = 4(1 + s_i s_j)/5$. Intuitively, the restriction limits the informativeness of s_i concerning s_j relative to its impact on bidder i's own payoff. The supporting analysis is essentially identical to the preceding case and is omitted.

Remark 4. A closed-form expression for $\bar{b}(\cdot)$ is unavailable, but the preceding analysis suggests how to approximate it numerically. First, identify a critical point in (8) through which the function must pass. The stable manifolds of system (8) approach this point with a slope equal to that of the negative eigenvector from the system's Jacobian matrix. An approximate linearized solution can be defined locally at the critical point. This solution can be extended to the domain's remainder with standard numerical methods.

Remark 5. Allowing for $N > 2$ bidders requires amending (4), the expression for an agent's expected payoff. Defeating an opponent with a low value-signal or a low budget implies different conclusions concerning the item's value. Each possibility needs to be accounted for on an opponent-by-opponent basis. Kotowski and Li (2014a) present this extension for the all-pay auction with interdependent values. The equilibrium with N bidders in that setting is qualitatively-similar to the two-bidder case. Che and Gale (1998) allow for N bidders and study canonical equilibria in the private-values case.

[14] These assumptions follow Milgrom and Weber (1982) and are standard in the literature.



(a) Level sets of $\beta(s, w)$. The discontinuity occurs along the dashed curve.

(b) Three dimensional sketch of $\beta(s, w)$.

Figure 4: Illustration of a non-canonical equilibrium strategy $\beta(s, w)$ when $\underline{w} > 0$.

3.2 Non-Canonical Equilibria

A canonical equilibrium may not exist in the absence of Assumption 1 or if $\underline{w} > 0$. To investigate this possibility, we focus on the case where $\underline{w} > 0$ as the logical parallel with Example 1 is easiest to appreciate. A canonical equilibrium may not exist when $\underline{w} > 0$ because the marginal return to a slightly higher bid changes appreciably at $\underline{w}$. A bid above $\underline{w}$ defeats a budget-constrained opponent while a bid below $\underline{w}$ does not. This change in a bid's marginal return is particularly pronounced in Example 1 due to a mass point in the budget distribution at $1/4$. As shown below, a mass point is not necessary to induce a similar jump discontinuity in the equilibrium strategy of a high-budget or unconstrained agent.

Consider the non-canonical strategy illustrated in Figure 4, which translates the intuition from Example 1 to a model with a continuous type-space. It has three parts. First, a bidder with a low value-signal bids less than $\underline{w}$. Second, a high-budget bidder increases his bid discontinuously when his value-signal is $\tilde{s}$. Third, a low-budget bidder cannot match the jump in a high-budget bidder's strategy at $\tilde{s}$. Competition is stratified and low-budget bidders with value-signals $s \in (\tilde{s}, \tilde{s}')$ compete only in a low range of bids. A novelty is that the distinction between high- and low-budget bidders is endogenous and changes with the agent's value-signal. In Figure 4(a), this boundary is given by the function $\tilde{\phi}(\cdot)$. If $w < \tilde{\phi}(s)$, the agent's budget is "low" (conditional on his value-signal) and he bids less than

$\underline{w}$; otherwise, his budget is "high" and he bids above $\underline{w}$.

While an equilibrium with the above characteristics *can* arise when $\underline{w} > 0$, its presence is not assured. For some parameters, a canonical equilibrium may apply instead. To characterize all possibilities, we extend Theorem 1 to allow $\underline{w} \geq 0$. Concurrently, we also strengthen Assumption 1 (see Remark 1 above) to simplify the resulting technical analysis.

Assumption 1′. *The c.d.f. of budget constraints $G(\cdot)$ is concave on $[\underline{w}, \overline{w}]$.*

Theorem 2. *Suppose Assumption 1′ holds and $\underline{w} \geq 0$. There exist constants $\tilde{s}$ and $\tilde{s}'$, increasing functions $\bar{b}\colon [0,1] \to [0, \overline{w}]$ and $\tilde{b}\colon [\tilde{s}, \tilde{s}'] \to [0, \underline{w}]$, and a nonincreasing function $\tilde{\phi}\colon [\tilde{s}, \tilde{s}'] \to [\underline{w}, \overline{w}]$ such that the first-price auction has a symmetric equilibrium $\beta(s,w)$ with the following properties:*

(a) For all $s \in [\tilde{s}, \tilde{s}')$ and $w < \tilde{\phi}(s)$, $\beta(s,w) = \tilde{b}(s)$. The function $\tilde{b}(\cdot)$ is the solution of

$$\tilde{b}'(s) = \Big(v(s,s) - \tilde{b}(s)\Big) \frac{G(\tilde{\phi}(s))h(s)}{H(\tilde{s}) + \int_{\tilde{s}}^{s} G(\tilde{\phi}(y))h(y)dy} \tag{10a}$$

satisfying the boundary condition $\tilde{b}(\tilde{s}) = \lim_{s \to \tilde{s}^-} \bar{b}(s)$.

(b) Otherwise, $\beta(s,w) = \min\{\bar{b}(s), w\}$. For all $s < \tilde{s}$, the function $\bar{b}(\cdot)$ is continuous, less than $\underline{w}$, and is the solution of

$$\bar{b}'(s) = \big(v(s,s) - \bar{b}(s)\big) \frac{h(s)}{H(s)} \tag{10b}$$

satisfying the boundary condition $\bar{b}(0) = 0$. For all $s > \tilde{s}$, the function $\bar{b}(\cdot)$ is continuous, greater than $\underline{w}$, and for a.e. $s > \tilde{s}$,

$$\bar{b}'(s) = \frac{\lambda(s)}{\gamma(\bar{b}(s))} \left(\frac{\bar{b}(s) - v(s,s)}{\eta(s,s) - \delta(\bar{b}(s), s)} \right). \tag{10c}$$

Remark 6. Recall that equation (5) characterizes the bid of an unconstrained agent when he bids below and above $\underline{w}$. Acknowledging the similarity between (5), (10b), and (10c), the bid of an unconstrained agent in Theorem 2 equals $\bar{b}(s)$, i.e. $\beta(s, \overline{w}) = \bar{b}(s)$.

Theorem 2 allows for both canonical and non-canonical equilibria. Which case applies depends on the economy's parameters. Two qualitatively distinct canonical equilibria are defined solely by part (b). The function $\bar{b}(\cdot)$ in a Type 1 canonical equilibrium is characterized only by (10c).[15] This class subsumes the preceding subsection's analysis. As in the preceding subsection, the function $\bar{b}(\cdot)$ coincides with the stable solutions of system (8), the function $b^*(\cdot)$. Given Assumption 1′, a necessary and sufficient condition for a Type 1 canonical equilibrium to exist is that $b^*(\cdot)$ exists and has domain $[0,1]$. Intuitively, such equilibria tend to arise when $\underline{w}$ is close to zero.

A Type 2 canonical equilibrium is characterized by (10b) and (10c).[16] For low values of s, $\bar{b}(s) = \alpha(s)$, the equilibrium bid in the absence of budget constraints and the solution to (10b).[17] If there exists a value $s_\alpha \in (0,1)$ such that $\alpha(s_\alpha) = \underline{w}$, then $\bar{b}(\cdot)$ makes a *continuous* transition into the range of bids above $\underline{w}$ at $\tilde{s} = s_\alpha$. For $s > \tilde{s}$, $\bar{b}'(s)$ is given by (10c). Given Assumption 1′, a necessary and sufficient condition for a Type 2 canonical equilibrium is the following: if $\alpha(s_\alpha) = \underline{w}$, then $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) \leq 0$. This condition must hold for (10c) to have an increasing solution at $(s_\alpha, \underline{w})$. Online Appendix B discusses Type 2 equilibria in further detail. Intuitively, such equilibria tend to arise when $\underline{w}$ is relatively large.

The third case corresponds to the non-canonical equilibrium sketched in Figure 4. In this case both parts (a) and (b) in Theorem 2 contribute to the equilibrium strategy's definition. In this subsection's remainder we outline this equilibrium's construction. The associated proofs are relegated to Appendix B. A numerical example is presented in Section 3.3.

Construction of a Non-Canonical Equilibrium Paralleling Section 3.1, the following discussion presumes the existence of a single critical point. The cases of no or multiple critical points are presented in Online Appendix B.

We first introduce an assumption (henceforth maintained) precluding Type 1 or Type 2 equilibria. It assures the applicability of the non-canonical case given Assumption 1′. Recall that $b^*(\cdot)$ defines the stable solutions to system (8) and $\alpha(\cdot)$ is the first-price auction's equilibrium strategy in the absence of budget constraints.

Assumption 2.

(a) The domain of $b^*(\cdot)$ *is* $[s_*, 1]$ *and* $s_* > 0$.

(b) There exists an $s_\alpha \in (0,1)$ *such that* $\alpha(s_\alpha) = \underline{w}$ *and* $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) > 0$.

[15] In this case, $\tilde{s} = 0$ and $\tilde{\phi}(s) = \underline{w}$.

[16] In this case, $\tilde{s} \in (0,1]$ and $\tilde{\phi}(s) = \underline{w}$.

[17] If $\alpha(s) \leq \underline{w}$ for all s, then $\tilde{s} = 1$ and a Type 2 equilibrium reduces to $\beta(s,w) = \alpha(s)$. Though nominally in the model, agents' budgets are always so large that they never bind in equilibrium.

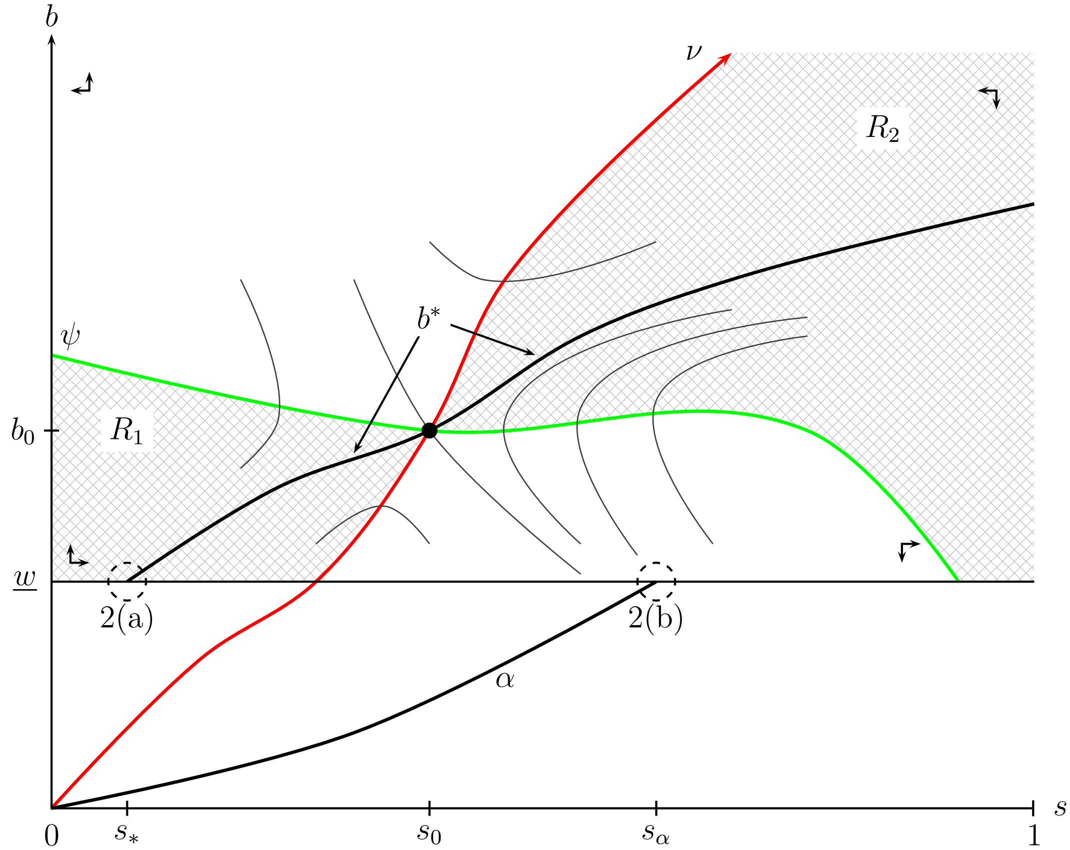


Figure 5: A portrait of system (8) when $\underline{w} > 0$. Assumptions 1′, 2(a), and 2(b) are satisfied.

Figure 5 illustrates a case satisfying Assumption 2. Assumption 2(a) says that $b^*(\cdot)$ is not defined for all s.[18] Assumption 2(b) implies (10c) does not have an increasing solution at $(s_\alpha, \underline{w})$. In the private values case, i.e. $v(s_i, s_j) = s_i$, it holds when $(1 - H(s_\alpha)) \int_0^{s_\alpha} H(z)/H(s_\alpha)^2 dz > 1/g(\underline{w})$. Thus, a sufficient condition for Assumption 2(b) to be satisfied is that the budget distribution is relatively concentrated near $\underline{w}$. In this light, Example 1 can be interpreted as a limiting case where $\underline{w} = 1/4$ is a mass point of the budget distribution.

Next, we explain how to identify $\tilde{s}$, $\bar{b}(\cdot)$, $\tilde{b}(\cdot)$, and $\tilde{\phi}(\cdot)$.[19] To identify $\tilde{s}$, we first introduce

[18]It can be verified that $b^*(\cdot)$ is defined near the right boundary given Assumption 1′. Thus, $s_* > 0$ is the only possible failure of the full domain assumption. In the absence of a critical point, Assumption 2 reduces to only point (b). See Online Appendix B.

[19]The value $\tilde{s}'$ is pinned down during the definition of $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. It is simply the largest value for which these functions' domain can be defined.

the function $\mu\colon [s_*, 1] \to [\underline{w}, \overline{w}]$ defined as

$$\mu(s) := \begin{cases} \min\{b^*(s), \psi(s)\} & \text{if } \psi(s) \neq \varnothing \\ \min\{b^*(s), \underline{w}\} & \text{if } \psi(s) = \varnothing \end{cases}.$$

Recall that $\psi(s)$ is the zero of the function $b \mapsto \eta(s, s) - \delta(b, s)$. The function $\mu(\cdot)$ is labeled in Figures 6 and 7. Through each point along this curve passes a solution of system (8) that can be extended as an increasing, continuous function to the boundary.[20] The value $\tilde{s}$ is a solution to

$$\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w), \tag{11}$$

where

$$\bar{U}(b|s, w) := \int_0^s \big(v(s, y) - b\big)h(y)dy + G(b)\int_s^1 \big(v(s, y) - b\big)h(y)dy. \tag{12}$$

In words, a bidder with value-signal $\tilde{s}$ is indifferent between the bids $\alpha(\tilde{s})$ and $\mu(\tilde{s})$. Lemma B.1 in Appendix B shows that (11) has a solution $\tilde{s} \in [s_*, s_\alpha]$ given the maintained assumptions.

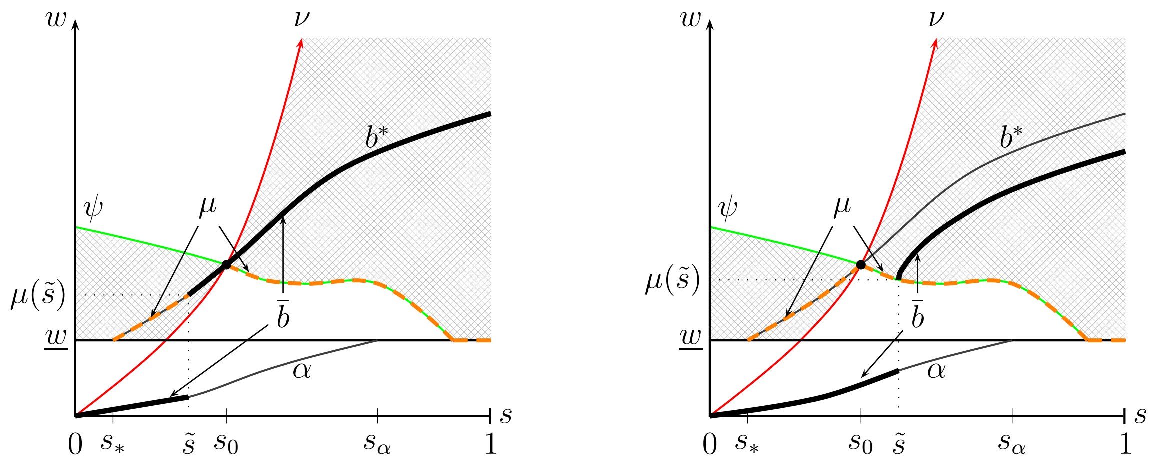
Given $\tilde{s}$, $\bar{b}(\cdot)$ is defined as follows. For $s < \tilde{s}$, $\bar{b}(s)$ solves (10b). Of course, this implies $\bar{b}(s) = \alpha(s)$ for $s < \tilde{s}$. For $s \geq \tilde{s}$, $\bar{b}(s)$ coincides with the increasing solution to system (8) passing through point $(\tilde{s}, \mu(\tilde{s}))$, which assures (10c) is satisfied. Figure 6 illustrates this definition in two representative cases. In Figure 6(a), $\mu(\tilde{s}) = b^*(\tilde{s})$ and $\bar{b}(s) = b^*(s)$ for all $s > \tilde{s}$. In Figure 6(b), $\mu(\tilde{s}) = \psi(\tilde{s})$ and $\bar{b}(s) < b^*(s)$ for all $s > \tilde{s}$. There is a jump discontinuity in $\bar{b}(\cdot)$ at $\tilde{s}$.

Given $\tilde{s}$ and $\bar{b}(\cdot)$, we can next pin down $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. The former solves (10a) subject to the boundary condition $\tilde{b}(\tilde{s}) = \lim_{s\to\tilde{s}^-} \bar{b}(s)$. Differential equation (10a) resembles (10b), but accounts for the reduced competition given the jump in a high-budget bidder's strategy. This is captured by the "$G(\tilde{\phi}(\cdot))$" weightings in the reversed hazard rate term.

The function $\tilde{\phi}(\cdot)$ defines the boundary between high- and low-budget bidders along which $\beta(s, w)$ is discontinuous. Accordingly, it is defined by an indifference condition. A type-$(s, \tilde{\phi}(s))$ bidder is indifferent between the bids $\tilde{b}(s)$ and $\tilde{\phi}(s)$. Thus,

$$\tilde{U}_{\tilde{\phi}}(\tilde{b}(s)|s, w) = \tilde{U}_{\tilde{\phi}}(\tilde{\phi}(s)|s, w) \tag{13}$$

[20]This solution may coincide with $b^*(s)$ and pass through the critical point.



(a) Definition of $\bar{b}(\cdot)$ when $\mu(\tilde{s}) = b^*(\tilde{s})$. (b) Definition of $\bar{b}(\cdot)$ when $\mu(\tilde{s}) = \psi(\tilde{s})$.

Figure 6: Identification of $\bar{b}(\cdot)$ when $\underline{w} > 0$.

for all $s \in [\tilde{s}, \tilde{s}']$ where

$$\tilde{U}_\varphi(b|s,w) := \int_0^{\tilde{s}} \big(v(s,y) - b\big)h(y)dy + \int_{\tilde{s}}^{s} \big(v(s,y) - b\big)G(\varphi(y))h(y)dy + G(b)\int_s^1 \big(v(s,y) - b\big)h(y)dy. \tag{14}$$

A subtle complication is that the definitions of $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$ reference one another. Thus, a fixed point argument is needed to show that these functions exist with the stated properties.

To reinforce intuition for $\tilde{\phi}(\cdot)$, recall Example 1. In that case, a high-budget bidder's strategy is discontinuous in a neighborhood of $\tilde{s}$. A low-budget agent bid less than 1/4 when $s \in (\tilde{s}, \tilde{s}')$, but his bid increases to 1/4 once $s > \tilde{s}'$. The function $\tilde{\phi}(\cdot)$ generalizes this sequence of jump discontinuities to a continuum type space.

Verification that Theorem 2 describes an equilibrium mimics the proof of Theorem 1 and is relegated to Appendix B.

Remark 7. Absence Assumption 1′, the auction's equilibrium will depend on the model's specifics. When the function $b \mapsto \eta(s,s) - \delta(b,s)$ crosses zero multiple times, some crossings occur at values where the budget distribution is (relatively) concentrated. Around these values, a bidder has an incentive to increase his bid substantially as the marginal returns to a slightly higher bid rise significantly. Jump discontinuities around such values allow $\bar{b}(s)$, the bid of an unconstrained bidder, to "skip around" regions where (10c) lacks increasing solutions. Multiple discontinuities are possible if $G(\cdot)$ is sufficiently irregular. Despite similar

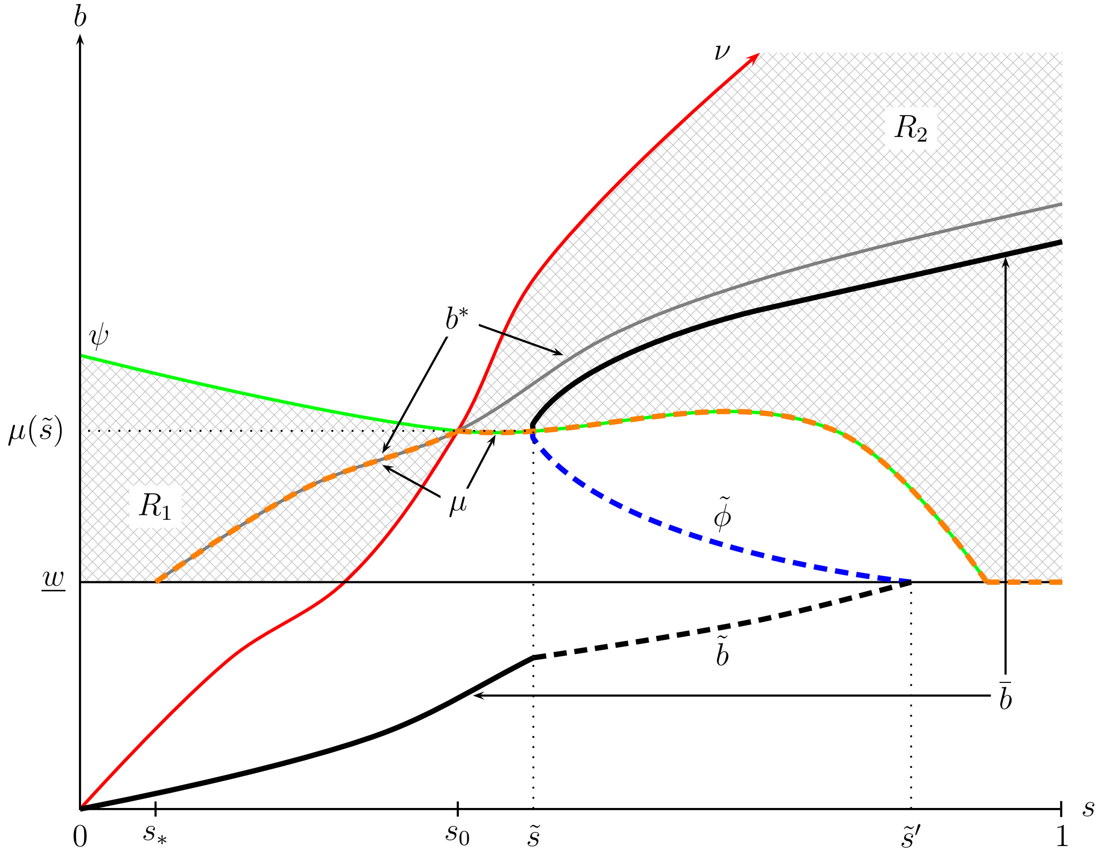


Figure 7: The functions $\bar{b}$, $\tilde{\phi}$, and $\tilde{b}$. The function $\bar{b}$ has a jump discontinuity at $\tilde{s}$.

intuition, identifying the analogues of $\tilde{\phi}(\cdot)$ and $\tilde{b}(\cdot)$ becomes challenging. Our proof does not extend immediately as it relies on the concavity of $G(\cdot)$ for this step.

3.3 An Example

An example can illustrate both canonical and non-canonical equilibria.

Example 2. Suppose $v(s_i, s_j) = s_i + s_j$ and value-signals are uniformly distributed on the unit interval. In the absence of budget constraints, $\alpha(s) = s$ is the symmetric equilibrium strategy. Suppose budgets are distributed uniformly on $[\underline{w}, 2]$ and $\underline{w} = 0$. Figure 8(a) shows $\bar{b}(s)$, $\psi(s)$, and $\nu(s)$. There is one critical point at $s_0 \approx 0.105$. Representative solutions from the associated plane-autonomous system and $\alpha(s)$ are graphed for context. Since values are interdependent, even a bidder with value-signal $s = 0$ bids a positive amount, i.e. $\bar{b}(0) > 0$.

Assumption 2 holds when $0.15 < \underline{w} < 0.27$; thus, a non-canonical equilibrium exists. Figure 8(b) illustrates $\bar{b}(s)$, $\psi(s)$, $\nu(s)$, $\tilde{\phi}(s)$, and $\tilde{b}(s)$ when $\underline{w} = 0.2$. The functions $\bar{b}(s)$ and $\alpha(s)$ coincide when s is small, but $\bar{b}(s)$ jumps up at $\tilde{s} \approx 0.181$. A high-budget bidder with that value-signal increases his bid from 0.181 to 0.286—an increase of 58 percent! Thus, an agent's strategic response to an opponent's possible budget constraint can be quantitatively large. When $\underline{w} > 0.27$, the equilibrium is again canonical, but of the Type 2 variety. The jump discontinuity in $\bar{b}(s)$ disappears.

To highlight the empirical implications, Figures 8(c) and 8(d) present histograms approximating the equilibrium bid distributions. In the canonical case, the peak occurs around $\bar{b}(0) \approx 0.12$. All bids below this value equal the bidder's budget. In the non-canonical case, the equilibrium bid distribution has few bids around $\underline{w} = 0.2$. The equilibrium bid distribution would be uniform in the absence of budget constraints.

3.4 Discussion

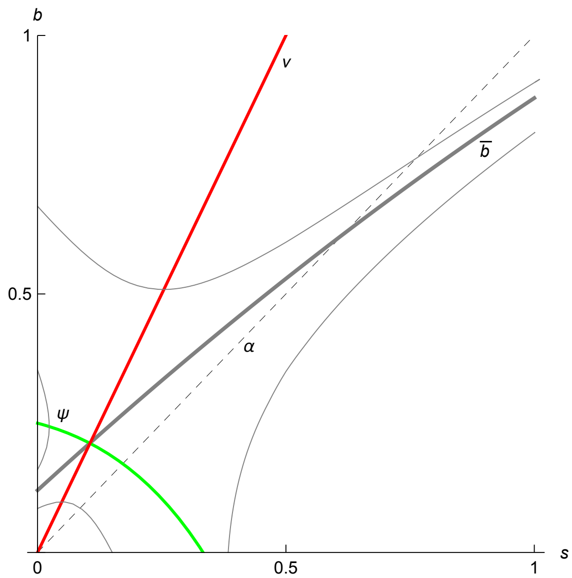
The Second-Price Auction It is instructive to compare equilibrium bidding in first- and second-price auctions. In the case of private values, i.e. $v(s_i, s_j) = s_i$, the second-price auction has an equilibrium where each type-(s, w) agent bids $\min\{s, w\}$ (Che and Gale, 1998). Fang and Parreiras (2002) examine the second-price auction with interdependent values in a two-bidder setting. They identify an equilibrium where each bidder's strategy is $\beta_{II}(s, w) = \min\{\bar{b}_{II}(s), w\}$ and $\bar{b}_{II}(\cdot)$ is an increasing function. When $\bar{b}_{II}(s) < \underline{w}$, $\bar{b}_{II}(s) = v(s, s)$, which is the equilibrium strategy identified by Milgrom and Weber (1982) in the absence of budget constraints. When $\bar{b}_{II}(s) \geq \underline{w}$, $\bar{b}_{II}(s)$ solves the differential equation[21]

$$\bar{b}'_{II}(s) = \frac{\lambda(s)}{\gamma(\bar{b}_{II}(s))} \left(\frac{\bar{b}_{II}(s) - v(s, s)}{\eta(s, s) - \bar{b}_{II}(s)} \right) \tag{15}$$

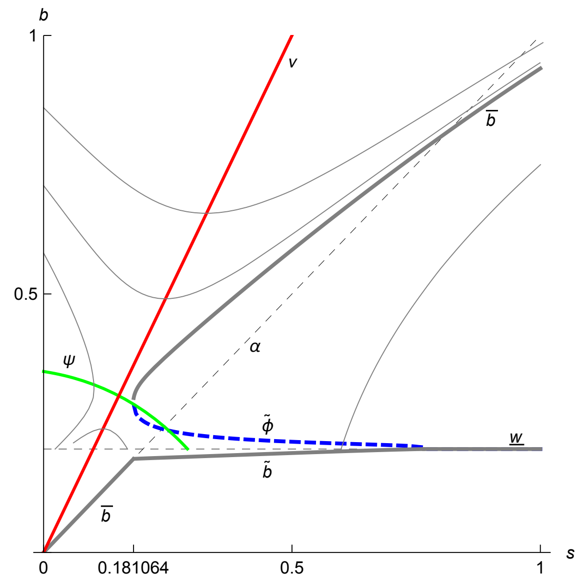
subject to the boundary condition $\bar{b}_{II}(1) = \overline{v}$. Since $\bar{b}_{II}(\cdot)$ is increasing, there exists a value $\tilde{s}_{II}$ such that $s < (>) \tilde{s}_{II} \implies \bar{b}_{II}(s) < (>) \underline{w}$. Generally, $\bar{b}_{II}(\cdot)$ may be discontinuous at $\tilde{s}_{II}$.

While the resemblance of (7) and (15) is striking, three differences between the first- and second-price auctions are noteworthy. First, the boundary condition for $\bar{b}_{II}(s)$ is predetermined. Second, the term "$\eta(s, s) - b$" appears in the denominator of (15) instead of "$\eta(s, s) - \delta(b, s)$." Third, the discontinuity that can arise in $\beta_{II}(s, w)$ at $\tilde{s}_{II}$ is qualitatively different from that identified in Section 3.2 above. There is no separation among high- and low-budget bidders in the second-price auction. A bidder with value-signal $s > \tilde{s}_{II}$ bids above $\underline{w}$ and a gap in the equilibrium bid distribution may arise. In contrast, the equilibrium bid

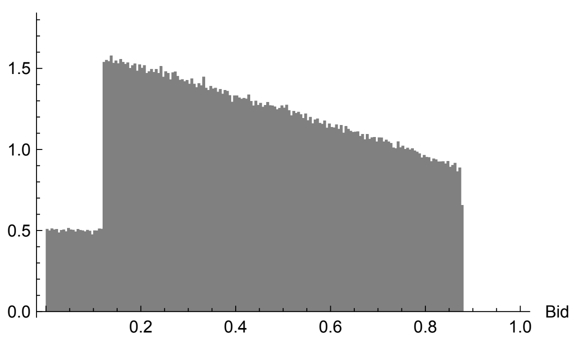
[21] Equation (15) restates equation (11) from Fang and Parreiras (2002). We specialize it to the case of independent types. Fang and Parreiras (2002) allow for affiliated values.



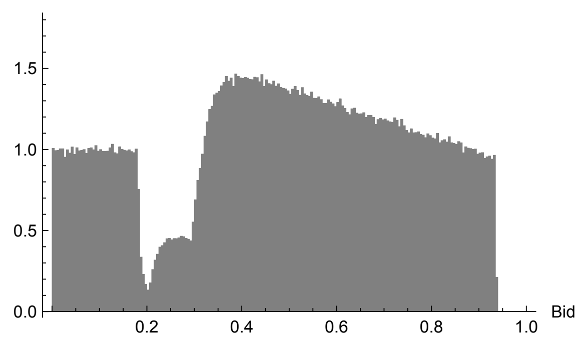
(a) Characterization of the equilibrium strategy when $w_i \overset{i.i.d.}{\sim} U[0, 2]$.



(b) Characterization of the equilibrium strategy when $w_i \overset{i.i.d.}{\sim} U[0.2, 2]$.



(c) Distribution of equilibrium bids when $w_i \overset{i.i.d.}{\sim} U[0, 2]$. Histogram of 1 million simulated bids (rescaled to unit area).



(d) Distribution of equilibrium bids when $w_i \overset{i.i.d.}{\sim} U[0.2, 2]$. Histogram of 1 million simulated bids (rescaled to unit area).

Figure 8: Equilibrium bidding in Example 2.

distribution in the first-price auction has connected support, even in a non-canonical equilibrium. This difference is due to the distinct pricing mechanisms. In the second-price auction, an agent bidding infinitesimally above the gap has no reason to reduce his bid. His probability of winning and his expected payment are unchanged if he bids a bit less. In the first-price auction, an agent bidding infinitesimally above a (hypothetical) gap can significantly reduce his expected payment without appreciably reducing his probability of winning. As a result, a gap in the equilibrium bid distribution cannot arise.[22]

Che and Gale (1998, 2006) show that the first-price auction revenue-dominates the second-price auction in an independent private values setting with financial constraints. Milgrom and Weber (1982) show that the first- and second-price auctions are revenue equivalent in the absence of budget constraints when valuations are interdependent but agents' types are independent.[23] They also prove that the second-price auction is revenue-superior when agents' types are affiliated. Taken together, these results imply that no revenue ranking exists between first- and second-price auctions allowing for both budget constraints and affiliated values.

Equilibrium Non-Uniqueness McAdams (2007) shows that the first-price auction has a unique monotone pure strategy equilibrium when bidders are ex ante symmetric, types are affiliated, and values are interdependent. This conclusion does not extend to a setting with budget constraints. Recall that in a non-canonical equilibrium, an agent discontinuously increases his bid at some value-signal $\tilde{s}$. He does so conjecturing the other bidder does likewise. In equilibrium, of course, this conjecture is correct but the construction introduces an implicit coordination problem. Different values at which the common bidding strategy "jumps up" may be compatible with different equilibria, as illustrated by the following example.

Example 3. Consider a first-price auction with two bidders.[24] Given (s_i, s_j), each bidder's valuation is $v(s_i, s_j) = s_i + s_j$. Each bidder's value-signal is independently distributed according to the c.d.f. $H(s) = \sqrt{s}$ on $[0, 1]$. In the absence of budget constraints, $\alpha(s) = 2s/3$ is the symmetric equilibrium bidding strategy (Milgrom and Weber, 1982).

[22]This reasoning does not apply when there are atoms in the equilibrium bid distribution, as illustrated by Example 1. In the example, the bid $1/4$ is placed with positive probability in equilibrium. An agent bidding $1/4$ would like to bid more but cannot due to his budget constraint. A bid slightly less than $1/4$, in contrast, leads to a *discrete* fall in the probability of winning and is therefore not beneficial. A gap in the equilibrium bid distribution just below $1/4$ results.

[23]See also the discussion in Krishna (2002, Section 7.1).

[24]Calculations pertaining to this example are presented in Online Appendix A.

Now suppose bidders face a common budget constraint: $w_i = w_j = 1/3$ with probability 1.[25] There are now two symmetric equilibria with strategies

$$\beta_A(s) = \begin{cases} 2s/3 & \text{if } s \leq 1/9 \\ 1/3 & \text{if } s > 1/9 \end{cases} \qquad \text{and} \qquad \beta_B(s) = 1/3.$$

In the "β_A"-equilibrium, the agents adopt the no-budget-constraints equilibrium strategy when s is small. At $\tilde{s}_A = 1/9$ their bids increase discontinuously from $2/27$ to $1/3$. In the "β_B"-equilibrium, the threshold value-signal above which agents bid their entire budget is $\tilde{s}_B = 0$. The "β_B"-equilibrium is revenue superior.

Comparative Statics The countervailing incentives associated with budget constraints typically lead to ambiguous comparative statics. For example, consider a "tightening" of financial constraints where the budget distribution is skewed in the sense of likelihood ratio dominance toward a distribution favoring lower budgets.[26] In the second-price auction, this change inflates the bid of an "unconstrained" bidder (Fang and Parreiras, 2002). In the first-price auction, a bidder's response may go either way. There is now less incentive to bid aggressively because competition is reduced at higher bid levels. However, if budgets become more concentrated around particular values it is easier to exploit the budget dimension for strategic gain. On balance, a bidder may respond with a higher or lower equilibrium bid. We document these phenomena in an example in Online Appendix C.

Reserve Prices Setting the optimal reserve price with budget-constrained bidders is challenging. Reserve prices screen bidders on both value and budget dimensions. Expected revenues may fall if bidders are disproportionately screened along the latter as the probability of sale is particularly impacted. If agents are unlikely to be budget constrained, imposing the no-budget-constraints revenue-maximizing reserve price can be a sensible policy. For example, if in the model of Example 1 an agent has a low budget ($w = 1/4$) with probability $p < 0.317$, the revenue-maximizing reserve price is $1/2$. This is also the revenue-maximizing reserve price in the absence of budget constraints. Otherwise, the optimal reserve price is less than $1/4$.

[25] We interpret this example as a limiting case of the more general model developed above. Posit, for instance, that $\underline{w} = 1/3$ is the lower limit of the budget distribution's support and its density becomes highly concentrated around this value.

[26] Likelihood ratio dominance implies hazard-rate dominance and first-order stochastic dominance.

4 Asymmetric Auctions and "Monotone" Equilibria

All equilibria in Section 3 are in nondecreasing strategies. A bidder with a higher budget or value-signal bids more. A natural question concerns this relationship's robustness, especially if budgets interact with preferences. For instance, consider risk preferences. A plausible claim is that a high-budget bidder is less risk averse than a low-budget rival, all else equal. A high-budget bidder may want to leverage his strategic advantage by bidding more. However, his higher risk tolerance can simultaneously tempt him to bid less.[27] Together these conflicting interests can lead to an equilibrium that fails to be "monotone" in the term's traditional sense, as shown by the following example.

Example 4. Recall Example 1, but now suppose that a winning bidder of type (s_i, w_i) receives a payoff of $(s_i - b_i)^{w_i + 1/4}$. Payoffs following a loss are zero.[28] Admittedly stylized, these payoffs capture the idea that a high-budget bidder is less risk-averse. A high budget ($w_i = 3/4$) implies risk-neutrality. A low budget ($w_i = 1/4$) implies strict risk-aversion. Figure 9 sketches this auction's symmetric equilibrium strategy,

$$\beta(s_i, 1/4) = \begin{cases} 2s_i/3 & \text{if } s_i \in [0, \tilde{s}''] \\ \frac{16s_i^3 + 24(\sqrt{11}-3)s_i^2 + 33(19\sqrt{11}-63)}{24(s_i + \sqrt{11}-3)^2} & \text{if } s_i \in (\tilde{s}'', \tilde{s}'] \\ 1/4 & \text{if } s_i \in (\tilde{s}', 1] \end{cases}$$

$$\beta(s_i, 3/4) = \begin{cases} s_i/2 & \text{if } s_i \in [0, \tilde{s}] \\ \frac{2s_i^2 + 13\sqrt{11}-42}{4(s_i+1)} & \text{if } s_i \in (\tilde{s}, 1] \end{cases}$$

where $\tilde{s} = \sqrt{11} - 3$, $\tilde{s}'' = 3(\sqrt{11} - 3)/4$, and $\tilde{s}' \approx 0.298$.[29] This strategy is neither canonical nor monotone. A high-budget bidder's risk tolerance explains his low bid when s_i is small. He exercises his strategic advantage to outbid a low-budget rival if s_i is large.

The absence of a monotone equilibrium clouds our understanding of the first-price auction with private budgets in general circumstances. Nevertheless, we can obtain some insight by noting that a systematic relationship still exists between types and bids. In Example 4, if s_i and w_i rise sufficiently and together, agent i will bid more. Reny (2011) showed that such regularities can be used to *re*define a nondecreasing strategy in games of incomplete

[27] Intuitively, the insurance provided by a higher bid is less valuable (Krishna, 2002, p. 38).
[28] Cox et al. (1988) consider a setting with a similar utility function, though without budget constraints.
[29] We verify that the defined strategy is an equilibrium in Online Appendix A.

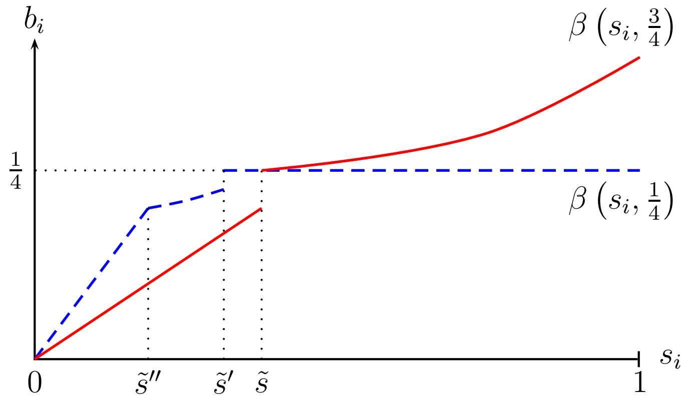


Figure 9: The equilibrium strategy in Example 4. Figure not to scale.

information. The idea is to realign changes in agents' types with changes in their equilibrium actions. This argument applies to the first-price auction with private budgets as well.

Generalizing our model, suppose there are N bidders, each with a private type $(s_i, w_i) \in [\underline{s}_i, \overline{s}_i] \times [\underline{w}_i, \overline{w}_i]$. Now, let $s = (s_1, \dots, s_N)$ be the vector of value-signals and s_{-i} be the vector s with s_i removed. Define w and w_{-i} analogously. Each bidder i has a set of valid bids $\mathcal{B}_i \subset \mathbb{R}$, which subsumes a reserve price $r_i \geq 0$ and includes a "null bid" $\ell_i < \min\{\underline{w}, r_i\}$ guaranteeing a loss in the auction.[30] The function $\beta_i \colon [\underline{s}_i, \overline{s}_i] \times [\underline{w}_i, \overline{w}_i] \to \mathcal{B}_i$ is an admissible (pure) strategy if $\beta_i(s_i, w_i) \leq w_i$ for all (s_i, w_i).

Suppose preferences satisfy the following conditions whenever $r_i \leq b_i \leq w_i$:

(i) If bidder i wins the auction with the bid b_i, his payoff is $\bar{u}_i(s_i, s_{-i}, w_i, b_i)$. The function $\bar{u}_i$ is bounded, nondecreasing in (s_i, s_{-i}, w_i), strictly decreasing in b_i, and differentiable. Moreover, $\partial \bar{u}_i / \partial s_i \geq \kappa_i > 0$.

(ii) If bidder i does not win the auction, his payoff is $\underline{u}_i(w_i)$. The function $\underline{u}_i$ is bounded, nondecreasing, and differentiable.

(iii) If $b_i \leq 0$, then $\bar{u}_i(s_i, s_{-i}, w_i, b_i) \geq \underline{u}_i(w_i)$. And, there exists $\bar{B}$ such that $\bar{u}_i(s_i, s_{-i}, w_i, b_i) < \underline{u}_i(w_i)$ for all $b_i > \bar{B}$.

(iv) For all $b_i \geq b_i' \geq r_i$, $\bar{u}_i(s_i, s_{-i}, w_i, b_i) - \bar{u}_i(s_i, s_{-i}, w_i, b_i')$ is nondecreasing in (s_i, s_{-i}, w_i).

(v) The following holds: $\inf \partial \bar{u}_i / \partial w_i \leq \sup \partial \underline{u}_i / \partial w_i < \infty$.

Remark 8. Section 3's model satisfies (i)–(v). Let $\bar{u}_i(s_i, s_{-i}, w_i, b_i) = v(s_i, s_{-i}) + w_i - b_i$ and $\underline{u}(w_i) = w_i$.

[30] We allow for bidder-specific reserve prices (Athey, 2001; Reny and Zamir, 2004; Kotowski, 2018).

Points (i)–(iv) adapt assumptions from Maskin and Riley (2000), Reny and Zamir (2004), and McAdams (2007), among others, to our setting. Points (i), (ii), and (iii) are standard. The lower bound on $\partial \bar{u}_i / \partial s_i$ in (i) is similar to Athey's (2001) Assumption A2(v). Point (iv) is an increasing-differences condition, common to the literature.[31]

Point (v) is new. It says that a bidder's money holdings (budget) and the item for purchase are not excessively strong complements—the minimal marginal utility of an extra dollar conditional on winning the auction is less than the maximal marginal utility of an extra dollar conditional on losing the auction.

Point (v) plays a role in the redefinition of a nondecreasing strategy. Reny (2011) pioneered this approach and, interestingly, we can draw on his analysis of a multi-unit auction with risk-averse bidders to study our problem. The key analogy is the following. An auction with private budgets can be interpreted as an unusual multi-item auction where the highest bidder wins the item for sale, as usual, and all bidders "win" a cash prize equal to their private budget. Noting this analogy, we follow Reny (2011) by introducing a new ordering of each agent's type-space, which we denote by $\geq_i$. Defining

$$\xi_i := \frac{\sup \frac{\partial \underline{u}_i}{\partial w_i} - \inf \frac{\partial \bar{u}_i}{\partial w_i}}{\inf \frac{\partial \bar{u}_i}{\partial s_i}}, \tag{16}$$

we say that $(s_i, w_i) \geq_i (s_i', w_i') \iff w_i \geq w_i'$ and $s_i - s_i' \geq \xi_i(w_i - w_i')$. Figure 10 sketches the greater- and less-than sets for type (s_i, w_i). We call the function β_i $\geq_i$-*nondecreasing* if $(s_i, w_i) \geq_i (s_i', w_i') \implies \beta_i(s_i, w_i) \geq \beta_i(s_i', w_i')$.

To interpret ξ_i and the $\geq_i$ ordering, it is helpful to temporarily ignore the "sup" and "inf" operations. In the quasilinear case, $\xi_i = 0$ because $\partial \underline{u}_i / \partial w_i = \partial \bar{u}_i / \partial w_i$. Hence, $\geq_i$ reduces to the standard coordinate-wise order of $\mathbb{R}^2$ and a $\geq_i$-nondecreasing strategy is nondecreasing in the usual sense. A large value for ξ_i can arise in two cases. If the numerator of (16) is large, then the marginal utility of an extra dollar is greater conditional on a loss than a win. All else equal, a bidder is more willing to bid *less* as w_i increases since the sting of a loss is less severe. A $\geq_i$-nondecreasing strategy does not mandate a higher bid in this case. Alternatively, if the denominator of (16) is small, then payoffs are relatively unresponsive to changes in a bidder's value-signal. Thus, a (relatively) large increase in s_i is necessary to assuredly counteract the (possible) countervailing incentives associated with an increase in w_i. To ensure that ξ_i is uniformly defined, the "sup" and "inf" operations pick out extreme

[31] See, for example, Assumption A.1(iv) and Remark 3 in Reny and Zamir (2004). The condition allows for risk aversion. For instance, it holds when $\bar{u}_i(s_i, s_{-i}, w_i, b_i) = V(s_i + w_i - b_i)$ and $V(\cdot)$ is concave.

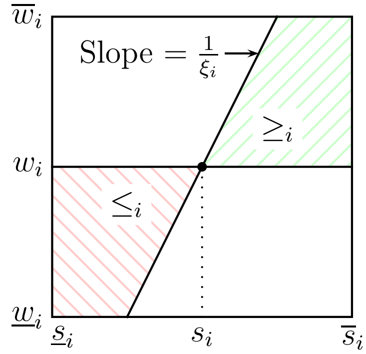


Figure 10: The $\geq_i$ ordering.

instances of the preceding cases. These help bound changes in a bidder's utility and allow us to confirm that $u_i(s_i, s_{-i}, w_i, b_i) \coloneqq \bar{u}_i(s_i, s_{-i}, w_i, b_i) - \underline{u}_i(w_i)$ is $\geq_i$-nondecreasing, a key step (Lemma C.1) in the proof of Theorem 3, which is stated below.

Concerning the information structure, we allow for affiliated value-signals, though qualified in the sense of Remark 3. Defining the terms

$$\begin{aligned} H(s_{-i}|s_i) &\coloneqq \int_{t_{-i} \leq s_{-i}} h(t_{-i}|s_i) dt_{-i}, & \bar{H}(s_{-i}|s_i) &\coloneqq \int_{t_{-i} \geq s_{-i}} h(t_{-i}|s_i) dt_{-i}, \\ \sigma(s_{-i}|s_i) &\coloneqq h(s_{-i}|s_i)/H(s_{-i}|s_i), & \lambda(s_{-i}|s_i) &\coloneqq h(s_{-i}|s_i)/\bar{H}(s_{-i}|s_i), \end{aligned}$$

we assume that the following conditions hold:

(vi) Value signals are affiliated with a strictly positive and bounded joint density $h(s)$. For a.e. s_{-i}, $(s_i, w_i) \geq_i (s_i', w_i')$, and $r_i \leq b_i \leq w_i'$,

(a) $u_i(s_i', s_{-i}, w_i', b_i) \geq 0 \implies u_i(s_i, s_{-i}, w_i, b_i)\lambda(s_{-i}|s_i) \geq u_i(s_i', s_{-i}, w_i', b_i)\lambda(s_{-i}|s_i')$; and,

(b) $u_i(s_i, s_{-i}, w_i, b_i) < 0 \implies u_i(s_i, s_{-i}, w_i, b_i)\sigma(s_{-i}|s_i) \geq u_i(s_i', s_{-i}, w_i', b_i)\sigma(s_{-i}|s_i')$.

(vii) Budgets are mutually independent and independent of value-signals with a strictly positive and bounded joint density. (The budget distribution may vary among bidders.)

The two requirements in point (vi) ensure that the direct influence of s_i on payoffs dominates its informational content concerning s_{-i}. The condition is satisfied by the cases cited in Remark 3 above, which introduced a similar restriction.

Theorem 3. *Consider a first-price auction satisfying conditions (i)–(vii) above. There exists an equilibrium in $\geq_i$-nondecreasing strategies if:*

(a) Each bidder's set of valid bids is finite and mutually disjoint; or,

(b) Values are private,[32] *each bidder i's set of valid bids is $\mathcal{B}_i = \{\ell_i\} \cup [r_i, \infty)$, and ties among high bidders are resolved with a uniform randomization.*

The proof of Theorem 3 involves verifying that the model satisfies the conditions in Reny (2011). We already mentioned the analogy between our model and his analysis of a multi-unit auction with risk-averse bidders. The key lemmas proceed similarly and quickly lead to part (a), which is the theorem's economically-salient conclusion. The restriction to finite and disjoint bid sets is a modeling device common in the auction literature, especially when types are multidimensional (for example, see McAdams, 2006).

The challenge in extending Theorem 3(b) to the interdependent-values case stems from economic, and not solely technical, considerations. It is instructive to see where standard arguments can fail. In a standard model, Athey (2001) and Reny and Zamir (2004) show that when an opponent adopts a nondecreasing strategy, a bidder benefits from bidding infinitesimally more to favorably resolve a hypothetical tie. This incentive is not assured in our setting due to the type-space's multidimensionality.[33] Defeating low-value-signal and low-budget opponents entails different inferences concerning the item's value. A bidder may hesitate to bid more when doing so disproportionately defeats low value-signal opponents.[34]

The preceding results contrast with those of Gentry et al. (2015), who have adapted Reny's (2011) analysis to confirm the existence of a monotone equilibrium in a private-values first-price auction where agents' degree of risk aversion and wealth are private information. Importantly, wealth is not associated with a constraint on bids in their model. Gentry et al. (2015) show that bids decline in wealth, due to agents' greater risk tolerance. This decline can occur in our setup, but the strategic option associated with budget constraints is a persistent countervailing force. A high-budget agent may wish to bid more to leverage his strategic advantage.

5 Concluding Remarks

We have examined the first-price auction under the assumption that agents face private budget constraints. Our analysis confirms that private budgets influence bidding in several

[32] That is, $u_i(s_i, s_{-i}, w_i, b_i) = u_i(s_i, s'_{-i}, w_i, b_i)$ for all s_{-i} and s'_{-i}.

[33] Reny and Zamir (2004) provide additional examples illustrating the complications introduced by multi-dimensional types in auctions.

[34] More generally, uniform tie-breaking is often inadequate to ensure equilibrium existence in bidding games with multidimensional private information (Araujo et al., 2008). Equilibrium existence can be restored by introducing endogenous tie-breaking rules (Simon and Zame, 1990; Jackson et al., 2002). In an earlier version of this study, a tie breaking rule proposed by Araujo and de Castro (2009) proved adequate for our model.

distinct ways. They cap participants' bids while simultaneously encouraging more aggressive bidding in equilibrium by some agents. Together, these responses can lead to nontrivial equilibrium outcomes. Agents may adopt discontinuous bidding strategies and equilibria may fail to be monotone in the traditional sense of the term. Two directions for further study are especially noteworthy. First, we have precluded many natural embellishments from our model—resale, collusion, credit and financing, among others. As noted in Section 2, several studies have probed these questions, but much work remains. And second, the empirical implications of private budgets in auctions remain poorly understood. The development of new models or identification strategies to tackle this question seems especially promising.

A Proofs Relating to Section 3.1

Lemma A.1. *Suppose Assumption 1 holds.*

(a) If $\psi(s) \neq \varnothing$*, then* $\psi(s) < \overline{v}$*.*

(b) Let $\epsilon > 0$*. If* $\psi(s') \neq \varnothing$ *and* $\psi(s) = \varnothing$ *for all* $s \in (s', s' + \epsilon)$*, then* $\psi(s') = \underline{w}$*.*

(c) Let $\epsilon > 0$*. If* $\psi(s') \neq \varnothing$ *and* $\psi(s) = \varnothing$ *for all* $s \in (s' - \epsilon, s')$*, then* $\psi(s') = \underline{w}$*.*

(d) If $\underline{w} = 0$*, then* $\psi(0) \neq \varnothing$*.*

Proof. Part (a) is immediate since $\delta(\overline{w}, s) > \overline{v} \geq \eta(s, s)$. The proofs of (b) and (c) are similar and we only prove (b). To derive a contradiction, suppose $\psi(s') = \hat{b} > \underline{w}$. By Assumption 1, there exists $\hat{b}' \in (\underline{w}, \hat{b})$ such that $\eta(s', s') - \delta(\hat{b}', s') > 0$. By continuity, there exists $\hat{s}' \in (s', s' + \epsilon)$ such that $\eta(s', s') - \delta(\hat{b}', \hat{s}') > 0$. Since $\delta(\overline{w}, \hat{s}') > \eta(\hat{s}', \hat{s}')$, we conclude that $\psi(s') \neq \varnothing$ by the intermediate value theorem—a contradiction. Part (d) is implied by Assumption 1 and the fact that $\eta(0, 0) - \delta(0, 0) \geq 0$. $\square$

Lemma A.2. *Suppose Assumption 1 holds. If* $\underline{w} = 0$*, then system* (8) *has at least one critical point.*

Proof. Recall that $\nu(0) = 0$, $\nu(1) = \overline{v}$, and $\psi(0) \neq \varnothing$. If $\psi(0) = 0$, then a critical point occurs at the origin. Instead, suppose $\psi(0) > 0$. There are two cases. First, if $\psi(s) \neq \varnothing$ for all s, then by the implicit function theorem $\psi(s)$ is continuous. Since $\psi(s) \leq \overline{v}$, the intermediate value theorem implies that $\psi(s_0) = \nu(s_0)$ at some $s_0 \in [0, 1]$. And second, if $\psi(s') = \varnothing$ for some s', then there exists an $s'' < s'$ such that $\psi(s)$ is a continuous function on the interval $[0, s'']$ and $\psi(s'') = \underline{w} = 0$. By the intermediate value theorem, there exists some $s_0 \leq s''$ such that $\psi(s_0) = \nu(s_0)$. $\square$

Lemma A.3. *Suppose Assumption 1 holds. If* (s_0, b_0) *is a critical point of system* (8)*, then it is (generically) either a node or a saddle point.*

Proof. We first show that the eigenvalues of the Jacobian matrix evaluated at a critical point (s_0, b_0),

$$J = \left. \begin{pmatrix} \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial b} \\ \frac{\partial \dot{b}}{\partial s} & \frac{\partial \dot{b}}{\partial b} \end{pmatrix} \right|_{(s_0, b_0)},$$

are real-valued. The eigenvalues will be real if

$$\left(\frac{\partial \dot{s}}{\partial s}-\frac{\partial \dot{b}}{\partial b}\right)^2+4\frac{\partial \dot{s}}{\partial b}\frac{\partial \dot{b}}{\partial s}$$

when evaluated at (s_0,b_0) is nonnegative. Thus,

$$\left.\frac{\partial \dot{s}}{\partial b}\right|_{(s_0,b_0)}=\gamma'(b_0)\underbrace{(\eta(s_0,s_0)-\delta(b_0,s_0))}_{0}+\gamma(b_0)\left(-\left.\frac{\partial \delta}{\partial b}\right|_{(s_0,b_0)}\right)=-\gamma(b_0)\left.\frac{\partial \delta}{\partial b}\right|_{(s_0,b_0)}.$$

We know that $\gamma(b_0)\neq 0$ and Assumption 1 implies that $\frac{\partial \delta}{\partial b}\big|_{(s_0,b_0)}>0$. Hence, $\frac{\partial \dot{s}}{\partial b}\big|_{(s_0,b_0)}<0$. Also,

$$\left.\frac{\partial \dot{b}}{\partial s}\right|_{(s_0,b_0)}=\left.\frac{\partial \lambda}{\partial s}\right|_{(s_0,b_0)}\underbrace{(b_0-v(s_0,s_0))}_{0}+\lambda(s_0)\left(-\left.\frac{dv(s,s)}{ds}\right|_{s=s_0}\right)<0.$$

Therefore, $4\frac{\partial \dot{s}}{\partial b}\frac{\partial \dot{b}}{\partial s}>0$ as needed.

Since the eigenvalues of J are real, by the Hartman-Grobman theorem, the critical point(s) will be a saddle or a node if the eigenvalues are non-zero. This is equivalent to J having a non-zero determinant at (s_0,b_0):

$$\det J\big|_{(s_0,b_0)}=\gamma(b_0)\lambda(s_0)\left(\left.\frac{d\eta(s,s)}{ds}\right|_{s=s_0}-\left.\frac{\partial \delta(b_0,s)}{\partial s}\right|_{s=s_0}-\left.\frac{\partial \delta(b,s_0)}{\partial b}\right|_{b=b_0}\left.\frac{d\nu(s)}{ds}\right|_{s=s_0}\right).$$

A perturbation of any of the distributions or of the valuation function is sufficient to ensure the preceding expression is non-zero. □

B Proof of Theorem 2

The proof of Theorem 2 has two parts. The first part concerns the construction of a non-canonical strategy and complements Section 3.2. Construction of a Type 1 canonical strategy proceeds as in Section 3.1. Type 2 canonical strategies are discussed in Online Appendix B. The second part verifies that the proposed strategy is a symmetric equilibrium.

B.1 Construction of a Non-Canonical Equilibrium Strategy

Outline The construction of a non-canonical equilibrium strategy with a discontinuity proceeds in several steps. First, Lemma B.1 confirms the existence of the value $\tilde{s}$. Given $\tilde{s}$, the definition of $\bar{b}(\cdot)$ is provided in Section 3.2 and Online Appendix B in other cases of interest. Then, we examine the functions $\tilde{b}(\cdot)$ and $\tilde{\phi}(\cdot)$. These functions are defined with reference to a fixed point of a particular operator, which is introduced in Remark B.4. Definitions B.1–B.4 and Lemma B.2 introduce preliminary concepts used in this definition. Subsequently, Lemmas B.3–B.6 prove some properties of $\tilde{\phi}(\cdot)$. Throughout we assume that Assumptions 1′ and 2 hold.

Lemma B.1. *Consider* $\bar{U}(b|\tilde{s}, w)$ *defined in* (12)*. There exists* $\tilde{s} \in [s_*, s_\alpha]$ *such that* $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$*. Moreover,* $\mu(\tilde{s}) > \underline{w}$*.*

Proof. We first make two observations. Recall that by convention $G(b) = 0$ if $b < \underline{w}$.

(i) Since $G(\alpha(s_*)) = G(\mu(s_*)) = 0$ and $\alpha(s_*) \leq \mu(s_*)$, the inequality $\bar{U}(\alpha(s_*)|s_*, w) \geq \bar{U}(\mu(s_*)|s_*, w)$ follows immediately.

(ii) Assumptions 1′ and 2 imply that $\partial\bar{U}(b|s_\alpha, w)/\partial b = g(b)(1-H(s_\alpha))(\eta(s_\alpha, s_\alpha)-\delta(b, s_\alpha)) \geq 0$ for all $b \in [\alpha(s_\alpha), \mu(s_\alpha)]$. Hence, $\bar{U}(\alpha(s_\alpha)|s_\alpha, w) \leq \bar{U}(\mu(s_\alpha)|s_\alpha, w)$.

As $\bar{U}(\alpha(s)|s, w)$ and $\bar{U}(\mu(s)|s, w)$ are continuous functions of s, there exists $\tilde{s} \in [s_*, s_\alpha]$ such that $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$. Finally, if $\mu(\tilde{s}) = \underline{w}$, then $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$ only if $\alpha(\tilde{s}) = \underline{w}$, which implies $\tilde{s} = s_\alpha$. But if $\underline{w} = \mu(s_\alpha) = \alpha(s_\alpha)$, then Assumption 2(b) implies that $\eta(\tilde{s}, \tilde{s}) - \delta(\underline{w}, \tilde{s}) > 0$ while the definition of $\mu(\cdot)$ implies that $\eta(\tilde{s}, \tilde{s}) - \delta(\underline{w}, \tilde{s}) \leq 0$—a contradiction. Hence, $\mu(\tilde{s}) > \underline{w}$. ☐

Remark B.1. Henceforth, let $\tilde{s} \in [s_*, s_\alpha]$ be a fixed value such that $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\mu(\tilde{s})|\tilde{s}, w)$. To simplify notation, let $\tilde{\mu} := \mu(\tilde{s})$.

Remark B.2. Definitions B.1 and B.2, which follow below, reference the constants $K_1 := \overline{v} \cdot [\sup_{s \geq \tilde{s}} h(s)]/H(\tilde{s})$ and $\tilde{s}_1 := \tilde{s} + (\underline{w} - \alpha(\tilde{s}))/K_1$.

Definition B.1 (The set $\mathscr{B}$)**.** Let $\mathscr{B}$ be the set of all continuous, nondecreasing, functions with domain $[\tilde{s}, \tilde{s}_1]$ such that $b(\tilde{s}) = \alpha(\tilde{s})$ and $\forall\ s, s' \in [\tilde{s}, \tilde{s}_1]$ and $b(\cdot) \in \mathscr{B}$, $|b(s) - b(s')| \leq K_1|s - s'|$.

Definition B.2 (The set $\mathscr{F}$)**.** Let $\mathscr{F}$ be the set of all continuous, nondecreasing functions with domain $[\tilde{s}, \tilde{s}_1]$ such that $F(\tilde{s}) = H(\tilde{s})$ and $\forall\ s, s' \in [\tilde{s}, \tilde{s}_1]$ and $F(\cdot) \in \mathscr{F}$, $|F(s) - F(s')| \leq (\sup_{s \in [0,1]} h(s))|s - s'|$.

Definition B.3 (The function Δ)**.** For $b(\cdot) \in \mathscr{B}$, and $F(\cdot) \in \mathscr{F}$ let

$$\begin{aligned}\Delta(s, w|b, F) :=& \Big[G(w)(1 - H(s))(\eta(s, s) - w) - wF(s)\Big] \\ & - \Big[G(b(s))(1 - H(s))(\eta(s, s) - b(s)) - b(s)F(s)\Big].\end{aligned}$$

Lemma B.2. *Fix $b(\cdot) \in \mathscr{B}$ and $F(\cdot) \in \mathscr{F}$.*

(a) The function $w \mapsto \Delta(s, w|b, F)$ is strictly concave on $[\underline{w}, \overline{w}]$.

(b) If $b(s) < \underline{w}$, then $\Delta(s, \underline{w}|b, F) < 0$.

(c) $\Delta(\tilde{s}, \tilde{\mu}|b, F) = 0$ and $\partial\Delta(\tilde{s}, w|b, F)/\partial w|_{w=\tilde{\mu}} \geq 0$.

Proof. Fix $b(\cdot) \in \mathscr{B}$ and $F(\cdot) \in \mathscr{F}$. To simplify notation, let $\Delta(s, w) := \Delta(s, w|b, F)$.

(a) Assumption $1'$ says that $G(w)$ is concave. Thus, $G(w)(1 - H(s))(\eta(s, s) - w)$ is strictly concave in w. It follows that $G(w)(1 - H(s))(\eta(s, s) - w) - wF(s)$ is strictly concave; hence $w \mapsto \Delta(s, w|b, F)$ is also strictly concave.

(b) Since $b(s) < \underline{w}$ for all $s \in [\tilde{s}, \tilde{s}_1)$, $\Delta(s, \underline{w}) = (b(s) - \underline{w})F(s) < 0$.

(c) Recall that $b(\cdot) \in \mathscr{B} \implies b(\tilde{s}) = \alpha(\tilde{s})$ and $F(\cdot) \in \mathscr{F} \implies F(\tilde{s}) = H(\tilde{s})$. Thus, by Lemma B.1, $\bar{U}(\alpha(\tilde{s})|\tilde{s}, w) = \bar{U}(\tilde{\mu}|\tilde{s}, w)$, which implies $\Delta(\tilde{s}, \tilde{\mu}) = 0$. Next, observe that

$$\frac{\partial}{\partial w}\Delta(s, w) = g(w)(1 - H(s))\underbrace{\left(\eta(s, s) - w - \frac{G(w)}{g(w)} - \frac{F(s)}{g(w)(1 - H(s))}\right)}_{A(s,w)}. \qquad \text{(B.1)}$$

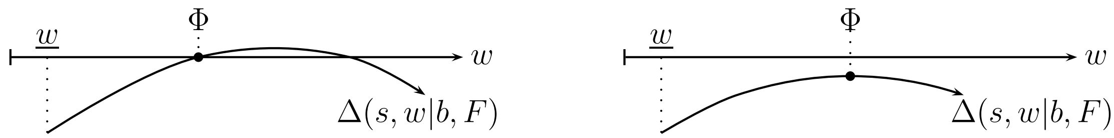
Since $\tilde{\mu} \leq \psi(\tilde{s})$, $A(\tilde{s}, \tilde{\mu}) \geq 0$, which implies $\partial\Delta(\tilde{s}, w)/\partial w|_{w=\tilde{\mu}} \geq 0$.

□

Definition B.4. The mapping $\Phi\colon [\tilde{s}, \tilde{s}_1] \times \mathscr{B} \times \mathscr{F} \to [\underline{w}, \overline{w}]$ is defined as

$$\Phi(s|b, F) := \min\left\{\operatorname*{argmin}_{w \in [\underline{w}, \overline{w}]} \Big|\Delta(s, w|b, F)\Big|\right\}.$$

Remark B.3. Figure B.1 illustrates the definition of Φ in two common cases. If $\Delta(s, w|b, F) = 0$ for some w, then Φ picks out the smallest solution (Figure B.1(a)). If $\Delta(s, w|b, F) < 0$ for all w, then Φ picks out its maximizer on $[\underline{w}, \overline{w}]$ (Figure B.1(b)).



(a) Case 1: Φ identifies the smallest solution of $\Delta(s, w|b, F) = 0$.

(b) Case 2: Φ identifies the maximizer of $\Delta(s, w|b, F)$ on the interval $[\underline{w}, \overline{w}]$.

Figure B.1: Definition of Φ.

Remark B.4 (Definition of $\tilde{\phi}(\cdot)$ and $\tilde{b}(\cdot)$). Define $\Lambda \colon \mathscr{B} \times \mathscr{F} \to \mathscr{B} \times \mathscr{F}$ as follows. If $(b, F) \in \mathscr{B} \times \mathscr{F}$, then $\Lambda(b, F) = (\check{b}, \check{F})$ where

$$\begin{aligned} \check{b}(s) &= \alpha(\tilde{s}) + \int_{\tilde{s}}^{s} \frac{(v(y,y) - b(y))G(\Phi(y|b,F))h(s)}{H(\tilde{s}) + \int_{\tilde{s}}^{s} G(\Phi(z|b,F))h(z)dz}\, dy, \text{ and} \\ \check{F}(s) &= H(\tilde{s}) + \int_{\tilde{s}}^{s} G(\Phi(y|b,F))h(y)dy. \end{aligned}$$

Λ is a continuous self-map defined on a compact, convex space. By Schauder's theorem, there exists a fixed point, $(\tilde{b}, \tilde{F}) = \Lambda(\tilde{b}, \tilde{F})$. Given such a fixed point $(\tilde{b}, \tilde{F})$, we define the function

$$\tilde{\phi}(s) := \Phi(s|\tilde{b}, \tilde{F})$$

and we simplify notation by henceforth writing

$$\tilde{\Delta}(s, w) := \Delta(s, w|\tilde{b}, \tilde{F}).$$

The preceding definitions, Lemma B.2, and Definition B.4 imply that

$$\tilde{b}'(s) = \frac{(v(s,s) - \tilde{b}(s))G(\tilde{\phi}(s))h(s)}{H(\tilde{s}) + \int_{\tilde{s}}^{s} G(\tilde{\phi}(y))h(y)dy} \qquad \text{and} \qquad \tilde{F}(s) = H(\tilde{s}) + \int_{\tilde{s}}^{s} G(\tilde{\phi}(y))h(y)dy.$$

Furthermore, $\tilde{b}(\tilde{s}) = \alpha(\tilde{s})$, $\tilde{b}(s) < \underline{w}$ for all $s \in [\tilde{s}, \tilde{s}_1)$, and $\tilde{\phi}(\tilde{s}) = \tilde{\mu} = \mu(\tilde{s})$.

Remark B.5. By substituting $\tilde{b}$ and $\tilde{\phi}$ we observe that

$$\tilde{\Delta}(s, \tilde{\phi}(s)) = \tilde{U}_{\tilde{\phi}}(\tilde{\phi}(s)|s, w) - \tilde{U}_{\tilde{\phi}}(\tilde{b}(s)|s, w)$$

where $\tilde{U}_{\varphi}(b|s, w)$ is defined in (14).

Lemma B.3. *For all* $s \in [\tilde{s}, \tilde{s}_1]$, $\tilde{\Delta}(s, \tilde{\phi}(s)) = 0$.

Proof. Since $\tilde{\phi}(\tilde{s}) = \tilde{\mu}$ and $\tilde{b}(\tilde{s}) = \alpha(\tilde{s})$, Lemma B.2(c) implies that $\tilde{\Delta}(\tilde{s}, \tilde{\phi}(\tilde{s})) = 0$. Suppose there exists $\hat{s} \in [\tilde{s}, \tilde{s}_1)$ such that $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$ and for all $\epsilon > 0$ (sufficiently small), $\tilde{\Delta}(\hat{s} + \epsilon, \tilde{\phi}(\hat{s} + \epsilon)) < 0$. (By definition of $\tilde{\phi}(s)$, $\tilde{\Delta}(s, \tilde{\phi}(s))$ cannot be strictly greater than zero.) Thus, for $s \in (\hat{s}, \hat{s} + \epsilon)$, $\tilde{\phi}(s) = \Phi(s|\tilde{b}, \tilde{F})$ is the unique maximizer of $\tilde{\Delta}(s, \cdot)$ on the interval $[\underline{w}, \overline{w}]$. There are two cases.

Case 1. Suppose $\tilde{\phi}(s) > \underline{w}$. Differentiating $\tilde{\Delta}(s, \tilde{\phi}(s))$, substituting for $\tilde{F}(s)$ and $\tilde{b}'(s)$, and applying the envelope theorem yields

$$\frac{d}{ds}\tilde{\Delta}(s, \tilde{\phi}(s)) = G(\tilde{\phi}(s)) \int_s^1 \frac{\partial v(s,y)}{\partial s} h(y)dy - G(\tilde{b}(s)) \int_s^1 \frac{\partial v(s,y)}{\partial s} h(y)dy.$$

Recall that $\tilde{\phi}(s) > \tilde{b}(s)$. And so $d\tilde{\Delta}(s, \tilde{\phi}(s))/ds > 0$ for $s \in (\hat{s}, \hat{s} + \epsilon)$. Since $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$, this implies $\tilde{\Delta}(s, \tilde{\phi}(s)) > 0$—a contradiction since $\tilde{\Delta}(s, \tilde{\phi}(s))$ is nonpositive.

Case 2. Suppose $\tilde{\phi}(s) = \underline{w}$. By continuity, $\tilde{\phi}(\hat{s}) = \underline{w}$, which implies $\tilde{\Delta}(\hat{s}, \underline{w}) = (\tilde{b}(\hat{s}) - \underline{w})\tilde{F}(\hat{s}) < 0$ since $\tilde{b}(\hat{s}) < \underline{w}$. Thus, we have derived a contradiction.

The above cases exhaust the possibilities; hence, for all $s \in [\tilde{s}, \tilde{s}_1)$, $\tilde{\Delta}(s, \tilde{\phi}(s)) = 0$. By continuity, the preceding conclusion applies to the boundary $s = \tilde{s}_1$ as well. □

Lemma B.4. *The function $\tilde{\phi}(s)$ is decreasing.*

Proof. Let $(\hat{s}, \hat{s}') \subset [\tilde{s}, \tilde{s}_1]$. We know that $\tilde{\Delta}(\hat{s}, \tilde{\phi}(\hat{s})) = 0$. Holding $\tilde{\phi}(\hat{s})$ fixed,

$$\frac{\partial}{\partial s}\tilde{\Delta}(s, \tilde{\phi}(\hat{s})) = G(\tilde{\phi}(\hat{s})) \int_s^1 \frac{\partial v(s,y)}{\partial s} h(y)dy - G(\tilde{b}(s)) \int_s^1 \frac{\partial v(s,y)}{\partial s} h(y)dy > 0.$$

This implies that $\tilde{\Delta}(\hat{s}', \tilde{\phi}(\hat{s})) > 0$. Thus, $\tilde{\phi}(\hat{s}')$ must be the smallest value that solves $\tilde{\Delta}(\hat{s}', \tilde{\phi}(\hat{s}')) = 0$. Since $\tilde{\Delta}(s, \cdot)$ is strictly concave, $\tilde{\phi}(\hat{s}') < \tilde{\phi}(\hat{s})$. □

Lemma B.5. *Suppose $\tilde{\phi}(s) > \underline{w}$. Then, $\partial\tilde{\Delta}(s, w)/\partial w|_{w=\tilde{\phi}(s)} = \partial\tilde{U}_{\tilde{\phi}}(b|s, w)/\partial b|_{b=\tilde{\phi}(s)} \geq 0$.*

Proof. The equality follows immediately from Remark B.5. That the derivative is greater than zero when evaluated at $\tilde{\phi}(s)$ follows from the definition of $\tilde{\phi}(s)$. The value $\tilde{\phi}(s)$ is either the smallest solution of $\phi \mapsto \tilde{\Delta}(s, \phi)$ or this function's maximizer. As the function is concave, its derivative must be nonnegative at $\tilde{\phi}(s)$ when $\tilde{\phi}(s) > \underline{w}$. □

Remark B.6 (Definition of $\tilde{s}'$). Above we defined $\tilde{b}$, $\tilde{F}$ and $\tilde{\phi}$ on the interval $[\tilde{s}, \tilde{s}_1]$. If $\tilde{b}(\tilde{s}_1) < \underline{w}$ then the preceding argument can be repeated inductively with $\tilde{s}_1$ replacing $\tilde{s}$, $\tilde{b}(\tilde{s}_1)$ replacing

$\alpha(\tilde{s})$, $\tilde{\phi}(\tilde{s}_1)$ replacing $\tilde{\mu} = \mu(\tilde{s})$, and $\tilde{F}(\tilde{s}_1)$ replacing $H(\tilde{s})$, etc. Let $[\tilde{s}, \tilde{s}']$ be the maximal interval on which the functions $\tilde{b}$, $\tilde{F}$ and $\tilde{\phi}$ can be defined inductively in this manner. The construction stops only in the limit. As the functions $\tilde{b}$, $\tilde{F}$ and $\tilde{\phi}$ are monotone, they can be extended to $\tilde{s}'$ by taking the appropriate limit from the left.

Lemma B.6. *Consider $\tilde{b}\colon [\tilde{s}, \tilde{s}'] \to \mathbb{R}$ and $\tilde{\phi}\colon [\tilde{s}, \tilde{s}'] \to [\underline{w}, \overline{w}]$ as defined in Remark B.6. Then, $\tilde{b}(\tilde{s}') = \underline{w} = \tilde{\phi}(\tilde{s}')$.*

Proof. If $\tilde{b}'(\tilde{s}') = \underline{w}$, then $\tilde{\Delta}(\tilde{s}', \tilde{\phi}(\tilde{s}')) = G(\tilde{\phi}(\tilde{s}'))(1-H(\tilde{s}'))(\eta(\tilde{s}', \tilde{s}')-\tilde{\phi}(\tilde{s}'))+(\underline{w}-\tilde{\phi}(\tilde{s}'))\tilde{F}(\tilde{s}')$. Recalling the definition of $\tilde{\phi}(\cdot)$ and that $\tilde{\Delta}(\tilde{s}', \tilde{\phi}(\tilde{s}')) = 0$, we conclude that $\tilde{\phi}(\tilde{s}') = \underline{w}$. $\square$

B.2 Verification of Equilibrium

Let β be the strategy described in Theorem 2. We will verify that an agent does not have an incentive to deviate to an alternative feasible bid in the range of β given that the other agent bids according to β. In a type 1 canonical equilibrium, the proof of Theorem 1 applies. In a type 2 canonical equilibrium, the range of β equals the range of $\bar{b}(\cdot)$ and the argument below applies.[35] Finally, in a non-canonical equilibrium, the range of β is characterized by four segments: (a) the function $\bar{b}(s)$ for $s < \tilde{s}$, (b) the function $\tilde{b}(s)$ for $s \in [\tilde{s}, \tilde{s}')$, (c) the function $\tilde{\phi}(s)$ for $s \in [\tilde{s}, \tilde{s}')$, and (d) the function $\bar{b}(s)$ for $s > \tilde{s}$. We divide our argument accordingly. Writing s^- and s^+ for left- and right-hand limits, respectively, we observe that $\bar{b}(\tilde{s}^-) = \tilde{b}(\tilde{s}^+) \le \tilde{b}(\tilde{s}'^-) = \underline{w} = \tilde{\phi}(\tilde{s}'^-) \le \tilde{\phi}(\tilde{s}^+) = \bar{b}(\tilde{s}^+)$.

Case 1. Consider a type-(s, w) bidder where $s < \tilde{s}$. This bidder's expected payoff when bidding $\beta(s, w) = \bar{b}(s)$ is $U(\bar{b}(s)|s, w) = \int_0^s (v(s, y) - \bar{b}(s))h(y)dy$.

(a) For all $s < \tilde{s}$, $\bar{b}(s) = \alpha(s)$, the equilibrium bidding strategy in a first-price auction without budget constraints. Therefore, this bidder has no profitable deviation to any alternative bid $\bar{b}(x)$ where $x < \tilde{s}$ (Milgrom and Weber, 1982). By continuity, the bid $\bar{b}(\tilde{s}^-)$ is also not a profitable deviation.

(b) If this agent bids $\tilde{b}(x)$ where $x \in [\tilde{s}, \tilde{s}')$, his payoff is

$$U(\tilde{b}(x)|s, w) = \int_0^{\tilde{s}} (v(s, y) - \tilde{b}(x))h(y)dy + \int_{\tilde{s}}^x (v(s, y) - \tilde{b}(x))G(\tilde{\phi}(y))h(y)dy. \tag{B.2}$$

[35] Only the cases involving a deviation to $\bar{b}(x)$ where $x \in [0, 1]$ are relevant.

Differentiating with respect to x and substituting $\tilde{b}'(x)$ as defined in (10a) gives $dU(\tilde{b}(x)|s,w)/dx = (v(s,x) - v(x,x))G(\tilde{\phi}(x))h(x)$. Since $s < \tilde{s} \le x$, $U(\tilde{b}(x)|s,w) \le U(\bar{b}(\tilde{s}^-)|s,w) \le U(\bar{b}(s)|s,w)$. Therefore, the bid $\tilde{b}(x)$ is not a profitable deviation.

(c) If this agent bids $\tilde{\phi}(x)$ for $x \in [\tilde{s}, \tilde{s}')$, his payoff is

$$\begin{aligned} U(\tilde{\phi}(x)|s,w) = &\int_0^{\tilde{s}} (v(s,y) - \tilde{\phi}(x))h(y)dy + \int_{\tilde{s}}^x (v(s,y) - \tilde{\phi}(x))G(\tilde{\phi}(y))h(y)dy \\ &+ G(\tilde{\phi}(x)) \int_x^1 (v(s,y) - \tilde{\phi}(x))h(y)dy. \end{aligned} \tag{B.3}$$

Subtracting (B.2) from (B.3),

$$\begin{aligned} &U(\tilde{\phi}(x)|s,w) - U(\tilde{b}(x)|s,w) \\ &= (\tilde{b}(x) - \tilde{\phi}(x)) \left(H(\tilde{s}) + \int_{\tilde{s}}^x G(\tilde{\phi}(y))h(y)dy \right) + G(\tilde{\phi}(x)) \int_x^1 (v(s,y) - \tilde{\phi}(x))h(y)dy \\ &\le (\tilde{b}(x) - \tilde{\phi}(x)) \left(H(\tilde{s}) + \int_{\tilde{s}}^x G(\tilde{\phi}(y))h(y)dy \right) + G(\tilde{\phi}(x)) \int_x^1 (v(x,y) - \tilde{\phi}(x))h(y)dy \\ &= U(\tilde{\phi}(x)|x,w) - U(\tilde{b}(x)|x,w) = 0. \end{aligned}$$

The inequality follows from $s \le x$. The final equality follows from the definition of $\tilde{\phi}$. See (13), (14), and Remark B.5 above. Thus, $U(\tilde{\phi}(x)|s,w) \le U(\tilde{b}(x)|s,w) \le U(\bar{b}(s)|s,w)$, where the final inequality is implied by sub-case (b) above.

(d) If this agent bids $\bar{b}(x)$ where $x \in (\tilde{s}, 1]$, his payoff is

$$U(\bar{b}(x)|s,w) = \int_0^x (v(s,y) - \bar{b}(x))h(y)dy + G(\bar{b}(x)) \int_x^1 (v(s,y) - \bar{b}(x))h(y)dy. \tag{B.4}$$

Since $s < x$, by the same argument as in the proof of Theorem 1, we know that $U(\bar{b}(x)|s,w) \le U(\bar{b}(\tilde{s}^+)|s,w)$. Since $\bar{b}(\tilde{s}^+) = \tilde{\phi}(\tilde{s})$, the proof of sub-case (c) implies that $U(\bar{b}(\tilde{s}^+)|s,w) \le U(\bar{b}(s)|s,w)$. Therefore, the bid $\bar{b}(x)$ where $x > \tilde{s}$ is not a profitable deviation.

Sub-cases (a)–(d) have shown that a bidder with value-signal $s < \tilde{s}$ cannot gain by deviating to another bid in the range of β.

Case 2. Consider a type-(s,w) bidder where $s \in (\tilde{s}, \tilde{s}')$ and $w < \tilde{\phi}(s)$. This bidder's expected payoff when bidding $\beta(s,w) = \tilde{b}(s)$ is

$$U(\tilde{b}(s)|s,w) = \int_0^{\tilde{s}} (v(s,y) - \tilde{b}(s))h(y)dy + \int_{\tilde{s}}^{s} (v(s,y) - \tilde{b}(s))G(\tilde{\phi}(y))h(y)dy.$$

Deviations to bids of (a) $\bar{b}(x)$ for $x < \tilde{s}$ or (b) $\tilde{b}(x)$ for $x \in [\tilde{s}, \tilde{s}')$ can be shown not to be profitable using an argument paralleling those presented in Cases 1(a) and 1(b) and Theorem 1. Thus, we only consider in the two remaining sub-cases, (c) and (d)

(c) If this agent bids $\tilde{\phi}(x)$, $x \in [\tilde{s}, \tilde{s}')$, his payoff is given by (B.3). If $x < s$, then $\tilde{\phi}(x) > w$. Hence, a deviation into this range of bids is not feasible for this bidder. Conversely, if $x > s$, then the argument in Case 1(c) establishes that $U(\tilde{\phi}(x)|s,w) \leq U(\tilde{b}(x)|s,w)$. Noting the preceding paragraph, we see that $U(\tilde{\phi}(x)|s,w) \leq U(\tilde{b}(s)|s,w)$.

(d) Since $w < \tilde{\phi}(s) \leq \bar{b}(\tilde{s}^+)$, the bid $\bar{b}(x)$ where $x > \tilde{s}$ is not feasible.

Case 3. Consider a type-(s,w) bidder where $s \in (\tilde{s}, 1]$ and $w \in [\tilde{\phi}(s), \tilde{\phi}(\tilde{s}^+)]$. There exists $s_w \in [\tilde{s}, s]$ such that $\beta(s,w) = \tilde{\phi}(s_w) = w$ and this bidder's expected payoff is

$$\begin{aligned} U(\tilde{\phi}(s_w)|s,w) = &\int_0^{\tilde{s}} (v(s,y) - \tilde{\phi}(s_w))h(y)dy + \int_{\tilde{s}}^{s_w} (v(s,y) - \tilde{\phi}(s_w))G(\tilde{\phi}(y))h(y)dy \\ &+ G(\tilde{\phi}(s_w)) \int_{s_w}^{1} (v(s,y) - \tilde{\phi}(s_w))h(y)dy. \end{aligned} \tag{B.5}$$

First, we consider case (c). Suppose this bidder places the bid $\tilde{\phi}(x)$ where $x \in [s_w, s]$. The agent's expected payoff at this bid is given by (B.5) with x replacing s_w. Consider the derivative,

$$\begin{aligned} &\frac{d}{dx}U(\tilde{\phi}(x)|s,w) \\ &= \tilde{\phi}'(x)\left[g(\tilde{\phi}(x))(1-H(x))\left(\eta(s,x) - \tilde{\phi}(x) - \frac{G(\tilde{\phi}(x))}{g(\tilde{\phi}(x))} - \frac{H(\tilde{s}) + \int_{\tilde{s}}^{x} G(\tilde{\phi}(y))h(y)dy}{g(\tilde{\phi}(x))(1-H(x))}\right)\right] \\ &\leq \tilde{\phi}'(x)\underbrace{\left[g(\tilde{\phi}(x))(1-H(x))\left(\eta(x,x) - \tilde{\phi}(x) - \frac{G(\tilde{\phi}(x))}{g(\tilde{\phi}(x))} - \frac{H(\tilde{s}) + \int_{\tilde{s}}^{x} G(\tilde{\phi}(y))h(y)dy}{g(\tilde{\phi}(x))(1-H(x))}\right)\right]}_{\frac{\partial}{\partial b}\tilde{U}_{\tilde{\phi}}(b|x,w)\big|_{b=\tilde{\phi}(x)}} \\ &\leq 0. \end{aligned}$$

The first inequality follows from $\tilde{\phi}'(x) \leq 0$ and $\eta(s,x) \geq \eta(x,x)$. The second inequality follows from Lemma B.5, which shows that the term in square brackets is nonnegative. Thus, each of these bids is dominated by $\tilde{\phi}(s_w) = w$.

If $s \leq \tilde{s}'$, then $U(\tilde{\phi}(s_w)|s,w) \geq U(\tilde{\phi}(s)|s,w) = U(\tilde{b}(s)|s,w)$ and the same argument as in Case 2 shows that this bidder has no profitable deviation to any bid below $\tilde{\phi}(s)$ as well.

Similarly, if $s \geq \tilde{s}'$, then $U(\tilde{\phi}(s_w)|s,w) \geq U(\underline{w}|s,w)$. Verifying that bids below $\tilde{b}(\tilde{s}') = \underline{w}$ are not profitable deviations proceeds similarly to the argument in Case 1, sub-cases (a) and (b). A deviation to the bid $\bar{b}(x)$ where $x > \tilde{s}$ is not feasible.

Case 4. Consider a type-(s,w) bidder where $s \geq \tilde{s}$ and $w \geq \tilde{\phi}(\tilde{s}'^{-})$.

The same argument as in the proof of Theorem 1 shows that this agent has no profitable deviation to any bid in the range of $\bar{b}(x)$ for $x \geq \tilde{s}$. In particular, this implies that he prefers $\beta(s,w)$ to the bid $\bar{b}(\tilde{s}^{-}) = \tilde{\phi}(\tilde{s})$. And so, the same argument as in Case 3 above shows that he cannot gain by deviating to a bid less than $\tilde{\phi}(\tilde{s})$.

The preceding cases have shown that a type (s,w) bidder does not have a feasible, profitable deviation from $\beta(s,w)$ given that the other bidder also adopts this strategy.

C Proof of Theorem 3

The proof of Theorem 3 follows a standard argument, building on Athey (2001) and Reny and Zamir (2004). We rely on Reny (2011) to confirm the existence of an equilibrium in $\geq_i$-nondecreasing strategies in a finite-action setting. This argument is similar to his study of a multi-unit auction. To simplify notation, in this appendix we let $g(w) := \prod_i g_i(w_i)$ be the joint p.f.d. of agents' budgets. Abusing notation, let $g(w_{-i}) := \prod_{j \neq i} g_j(w_j)$. We write $\theta_i = (s_i, w_i)$ for agent i's type and the joint p.d.f. of agents' types is $f(\theta) := h(s)g(w)$.

Define $\varphi_i(b_i, b_{-i})$ as the probability that bidder i wins the auction with bid b_i given that the profile of others' bids is b_{-i}. If b_i is the unique high bid, $\varphi_i(b_i, b_{-i}) = 1$. If b_i is strictly less than the highest bid or $b_i = \ell_i$, $\varphi_i(b_i, b_{-i}) = 0$. In the absence of ties, when others' strategies are β_{-i}, bidder i's expected payoff at bid b_i given $\theta_i = (s_i, w_i)$ is

$$U_i(b_i, \beta_{-i}|\theta_i) = \underline{u}_i(w_i) + \int_{\Theta_{-i}} \varphi_i(b_i, \beta_{-i}(\theta_{-i}))u_i(s, w_i, b_i)f(\theta_{-i}|\theta_i)d\theta_{-i}.$$

Bidder i's ex ante payoff is $U_i(\beta_i, \beta_{-i}) = \mathbb{E}[U_i(\beta_i(\theta_i), \beta_{-i}|\theta_i)]$. The profile $\beta^* = (\beta_i^*, \beta_{-i}^*)$ is a Nash equilibrium if for every i, $U_i(\beta_i^*, \beta_{-i}^*) \geq U_i(\beta_i, \beta_{-i}^*)$ for all admissible strategies β_i. When convenient, we use capital letters to refer to signals/types as random variables.

The following are preliminary lemmas. Lemma C.1 shows that $u_i(s_i, s_{-i}, w_i, b_i) \coloneqq \bar{u}_i(s_i, s_{-i}, w_i, b_i) - \underline{u}_i(w_i)$ is $\geq_i$-nondecreasing. Lemma C.2 verifies a single crossing condition.

Lemma C.1. *Let $r_i \leq b_i \leq w_i'$. If $(s_i, w_i) \geq_i (s_i', w_i')$, then $u_i(s_i, s_{-i}, w_i, b_i) \geq u_i(s_i', s_{-i}, w_i', b_i)$.*

Proof. Reny (2011) provides a similar argument in the context of a multi-unit auction with risk averse bidders. Since $s_i - s_i' \geq \xi_i(w_i - w_i')$ where $\xi_i = \left(\sup \frac{\partial \underline{u}_i}{\partial w_i} - \inf \frac{\partial \bar{u}_i}{\partial w_i}\right) / \inf \frac{\partial \bar{u}_i}{\partial s_i}$, we can establish the following inequalities:

$$\begin{aligned}
\bar{u}_i(s_i, s_{-i}, w_i, b_i) - \bar{u}_i(s_i', s_{-i}, w_i', b_i) &\geq \left[\inf \frac{\partial \bar{u}_i}{\partial s_i}\right](s_i - s_i') + \left[\inf \frac{\partial \bar{u}_i}{\partial w_i}\right](w_i - w_i') \\
&\geq \left[\inf \frac{\partial \bar{u}_i}{\partial s_i}\right]\xi_i(w_i - w_i') + \left[\inf \frac{\partial \bar{u}_i}{\partial w_i}\right](w_i - w_i') \\
&\geq \left[\sup \frac{\partial \underline{u}_i}{\partial w_i}\right](w_i - w_i') \\
&\geq \underline{u}_i(w_i) - \underline{u}_i(w_i').
\end{aligned}$$

Rearranging the final inequality gives the desired result. $\square$

Lemma C.2. *Let β_{-i} be a profile of $\geq_i$-nondecreasing strategies for bidders $j \neq i$. Let $\theta_i \geq_i \theta_i'$ and suppose the bids $b_i > b_i'$ tie with probability zero with the bids of the other agents.*

(a) $U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \geq 0 \implies U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i) \geq 0.$

(b) $U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i) \leq 0 \implies U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \leq 0.$

Proof. We prove part (a). Part (b) follows similarly. It is sufficient to show that $U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \leq U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i)$. First, define the following subsets of Θ_{-i}:

$$\begin{aligned}
A' &= \{(s_{-i}, w_{-i}) \mid \max_{j \neq i} \beta_j(s_j, w_j) \leq b_i'\}, \\
A &= \{(s_{-i}, w_{-i}) \mid \max_{j \neq i} \beta_j(s_j, w_j) \leq b_i\}, \\
\tilde{A} &= A \setminus A', \\
\tilde{A}^+ &= \{(s_{-i}, w_{-i}) \in \tilde{A} \mid u_i(s_i, s_{-i}, w_i, b_i) \geq 0\}, \\
\tilde{A}^- &= \{(s_{-i}, w_{-i}) \in \tilde{A} \mid u_i(s_i, s_{-i}, w_i, b_i) < 0\}.
\end{aligned}$$

Then,

$$
\begin{aligned}
0 &\leq U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \\
&= \underbrace{\int_{A'} \big(u_i(s_i', s_{-i}, w_i', b_i) - u_i(s_i', s_{-i}, w_i', b_i')\big) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i}}_{[1]} \\
&\quad + \underbrace{\int_{\tilde{A}^-} u_i(s_i', s_{-i}, w_i', b_i) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i}}_{[2]} \\
&\quad + \underbrace{\int_{\tilde{A}^+} u_i(s_i', s_{-i}, w_i', b_i) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i}}_{[3]}.
\end{aligned}
$$

Consider term [1]. By assumption $u_i(s_i, s_{-i}, w_i, b_i) - u_i(s_i, s_{-i}, w_i, b_i')$ is nondecreasing in s_{-i} and negative for all $(s_{-i}, w_{-i}) \in A'$. Since β_j is nondecreasing in s_j, $(s_{-i}, w_{-i}) \in A' \implies (s_{-i}', w_{-i}) \in A' \ \forall\ s_{-i}' \leq s_{-i}$. Thus, if $\mathbf{1}_{A'}(s_{-i}, w_{-i})$ is the indicator function for the set A', the function

$$
\chi(s_{-i}, w_{-i}|\theta_i) \coloneqq \big(u_i(s_i, s_{-i}, w_i, b_i) - u_i(s_i, s_{-i}, w_i, b_i')\big)\mathbf{1}_{A'}(s_{-i}, w_{-i})
$$

is nondecreasing in s_{-i} when w_{-i} is fixed. Hence,

$$
\begin{aligned}
&\int_{A'} \big(u_i(s_i', s_{-i}, w_i', b_i) - u_i(s_i', s_{-i}, w_i', b_i')\big) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&= \mathbb{E}\Big[\chi(S_{-i}, W_{-i}|\theta_i') \mid \theta_i'\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\left[\chi(S_{-i}, W_{-i}|\theta_i')|W_{-i} = w_{-i}\right] \mid \theta_i'\Big] \\
&\leq \mathbb{E}\Big[\mathbb{E}\left[\chi(S_{-i}, W_{-i}|\theta_i')|W_{-i} = w_{-i}\right] \mid \theta_i\Big] \\
&\leq \mathbb{E}\Big[\mathbb{E}\left[\chi(S_{-i}, W_{-i}|\theta_i)|W_{-i} = w_{-i}\right] \mid \theta_i\Big] \\
&= \int_{A'} \big(u_i(s_i, s_{-i}, w_i, b_i) - u_i(s_i, s_{-i}, w_i, b_i')\big) h(s_{-i}|s_i) g(w_{-i}) dw_{-i} ds_{-i}.
\end{aligned}
$$

Next, consider term [2]. First,

$$
\begin{aligned}
&\int_{\tilde{A}^-} u_i(s_i', s_{-i}, w_i', b_i) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&= \int_{\tilde{A}^-} u_i(s_i', s_{-i}, w_i', b_i) \frac{h(s_{-i}|s_i')}{H(s_{-i}|s_i')} H(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&\leq \int_{\tilde{A}^-} u_i(s_i, s_{-i}, w_i, b_i) \frac{h(s_{-i}|s_i)}{H(s_{-i}|s_i)} H(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&\leq \int_{\tilde{A}^-} u_i(s_i, s_{-i}, w_i, b_i) h(s_{-i}|s_i) g(w_{-i}) dw_{-i} ds_{-i}.
\end{aligned}
$$

The second inequality follows from $u(s_i, s_{-i}, w_i, b_i) \leq 0$ and $H(s_{-i}|s_i')/H(s_{-i}|s_i) \geq 1$.

Similar reasoning applies to term [3]:

$$
\begin{aligned}
&\int_{\tilde{A}^+} u_i(s_i', s_{-i}, w_i', b_i) h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&= \int_{\tilde{A}^+} \max\{u_i(s_i', s_{-i}, w_i', b_i), 0\} h(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&= \int_{\tilde{A}^+} \max\{u_i(s_i', s_{-i}, w_i', b_i), 0\} \frac{h(s_{-i}|s_i')}{\bar{H}(s_{-i}|s_i')} \bar{H}(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&\leq \int_{\tilde{A}^+} \max\{u_i(s_i, s_{-i}, w_i, b_i), 0\} \frac{h(s_{-i}|s_i)}{\bar{H}(s_{-i}|s_i)} \bar{H}(s_{-i}|s_i') g(w_{-i}) dw_{-i} ds_{-i} \\
&\leq \int_{\tilde{A}^+} \max\{u_i(s_i, s_{-i}, w_i, b_i), 0\} h(s_{-i}|s_i) g(w_{-i}) dw_{-i} ds_{-i} \\
&\leq \int_{\tilde{A}^+} u_i(s_i, s_{-i}, w_i, b_i) h(s_{-i}|s_i) g(w_{-i}) dw_{-i} ds_{-i}.
\end{aligned}
$$

The second inequality follows since $\bar{H}(s_{-i}|s_i')/\bar{H}(s_{-i}|s_i) \leq 1$. And, the third is true because $u_i(s_i, s_{-i}, w_i, b_i) \geq 0$. Together, the preceding cases imply that $0 \leq U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \leq U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i)$. $\square$

Next, consider a finite-action variant of the model incorporating three modifications. First, restrict each agent's set of admissible bids to a finite set $\mathcal{B}_i \subset \{\ell_i\} \cup [r_i, \infty)$ where $r_i \geq 0$ and $\ell_i < r_i$. Assume that $\ell_i \in \mathcal{B}_i$ and $\forall i \neq j$, $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$. Therefore, relevant ties among bidders are not possible. Second, modify each bidder's utility function so that he incurs a penalty if he bids more than this private budget irrespective of the auction's outcome.[36] Suppose that this penalty is sufficiently large to ensure that all bids exceeding

[36]Che and Gale (1998) suggest using a penalty to ensure adherence to the budget constraint. As a modeling device it ensures that each agent's strategy set is independent of his type.

w_i are strictly dominated by the bid ℓ_i for a type-(s_i, w_i) bidder. Third, let $\tilde{\epsilon} > 0$ be small and restrict each agent with a value-signal of $s_i \leq \underline{s}_i + \tilde{\epsilon}$ to bid ℓ_i. This technical restriction is used by Athey (2001) and Reny and Zamir (2004) and its role here is the same. (Similarly, below we let $\tilde{\epsilon} \to 0$.) Among other considerations, it ensures that all bids $b_i > \ell_i$ can win the auction with positive probability. All bids above $\bar{B}$ are now strictly dominated by the bid ℓ_i.

In this modified game, let $\rho_i(\theta_i|\beta_{-i})$ be the set of bidder i's best-response bids when his type is θ_i and others follow the strategy profile β_{-i}. Since $\mathcal{B}_i$ is finite, $\rho_i(\theta_i|\beta_{-i})$ is not empty. It is also bounded, $\max \rho_i(\theta_i|\beta_{-i}) \leq \max\{\bar{B}, w_i\}$.

Lemma C.3. *Consider the modified auction game described above. Fix a $\geq_i$-nondecreasing strategy profile β_{-i}. The set $\rho_i(\theta_i|\beta_{-i})$ is nondecreasing in the strong set order.*

Proof. Let $\theta_i \geq_i \theta_i'$ and let $b_i \in \rho_i(\theta_i|\beta_{-i})$ and $b_i' \in \rho_i(\theta_i'|\beta_{-i})$. If $b_i \geq b_i'$ for all such bids, we are done. Otherwise, suppose $b_i' > b_i$. Since $w_i \geq w_i' \geq b_i' > b_i$, both bids are feasible when a bidder has type θ_i or θ_i'. By Lemma C.2,

$$0 \leq U_i(b_i', \beta_{-i}|\theta_i') - U_i(b_i, \beta_{-i}|\theta_i') \leq U_i(b_i', \beta_{-i}|\theta_i) - U_i(b_i, \beta_{-i}|\theta_i) \implies b_i' \in \rho_i(\theta_i|\beta_{-i}),$$

and

$$0 \leq U_i(b_i, \beta_{-i}|\theta_i) - U_i(b_i', \beta_{-i}|\theta_i) \leq U_i(b_i, \beta_{-i}|\theta_i') - U_i(b_i', \beta_{-i}|\theta_i') \implies b_i \in \rho_i(\theta_i'|\beta_{-i}).$$

Hence, $\rho_i(\theta_i|\beta_{-i})$ is nondecreasing. □

Remark C.1. As the interim best reply correspondence is nondecreasing, player i has at least one nondecreasing best reply when others are following nondecreasing strategies (Topkis, 1998, Theorem 2.8.3).

Lemma C.4. *There exists an equilibrium in $\geq_i$-nondecreasing strategies in the modified auction game described above.*

Proof. The conclusion follows from Reny (2011) and specifically his Proposition 4.4, Theorems 4.1 and 4.3, and Remark 1. His conditions G.1–G.6 are satisfied by our game and are easy to verify. □

Lemma C.5 (Theorem 3(a))**.** *Consider the modified auction, where $\tilde{\epsilon} = 0$. There exists an equilibrium in $\geq_i$-nondecreasing strategies.*

Proof. Our proof builds on similar arguments provided by Athey (2001) and Reny and Zamir (2004), among others. The argument is simpler as ties are explicitly precluded.

Consider a sequence of auctions indexed by k where the measure of agents constrained to bid ℓ_i becomes arbitrarily small, i.e. $\tilde{\epsilon}^k \to 0$. Let β^k be the corresponding profile of equilibrium strategies. As each β_i^k is $\geq_i$-monotone there exists a strategy β_i such that $\beta_i^k(\theta_i) \to \beta_i(\theta_i)$ for all i and almost every θ_i. For each i, β_i is $\geq_i$-monotone. We will verify that β is an equilibrium when $\tilde{\epsilon} = 0$.

Suppose the contrary and that bidder i has a profitable deviation from β_i to $\hat{\beta}_i$ when β_{-i} is the strategy profile of the other bidders. Thus, $U_i(\hat{\beta}_i, \beta_{-i}) - U_i(\beta_i, \beta_{-i}) = \Delta > 0$. Since $\beta_i^k(\theta_i) \to \beta_i(\theta_i)$ point-wise, this convergence is uniform except possibly on a set of arbitrarily small measure (Egorov's Theorem). Choose $\varepsilon > 0$ sufficiently small such that $2\varepsilon \sup u_i(s, w_i, b) < \Delta/3$. Let Θ^ε be the set of measure (at most) ε where convergence need not be uniform. Next, consider

$$
\begin{aligned}
&U_i(\beta_i^k(\theta_i), \beta_{-i}^k|\theta_i) - U_i(\beta_i(\theta_i), \beta_{-i}|\theta_i) \\
&= \int_{\Theta_{-i}\setminus\Theta_{-i}^\varepsilon} \varphi_i(\beta_i^k(\theta_i), \beta_{-i}^k(\theta_{-i}))u_i(s, w_i, \beta_i^k(\theta_i))f(\theta_{-i}|\theta_i)d\theta_{-i} \\
&\quad - \int_{\Theta_{-i}\setminus\Theta_{-i}^\varepsilon} \varphi_i(\beta_i(\theta_i), \beta_{-i}(\theta_{-i}))u_i(s, w_i, \beta_i(\theta_i))f(\theta_{-i}|\theta_i)d\theta_{-i} \\
&\quad + \int_{\Theta_{-i}^\varepsilon} \varphi_i(\beta_i^k(\theta_i), \beta_{-i}^k(\theta_{-i}))u_i(s, w_i, \beta_i^k(\theta_i))f(\theta_{-i}|\theta_i)d\theta_{-i} \\
&\quad - \int_{\Theta_{-i}^\varepsilon} \varphi_i(\beta_i(\theta_i), \beta_{-i}(\theta_{-i}))u_i(s, w_i, \beta_i(\theta_i))f(\theta_{-i}|\theta_i)d\theta_{-i} \\
&< \int_{\Theta_{-i}\setminus\Theta_{-i}^\varepsilon} \left[\begin{array}{c} \varphi_i(\beta_i^k(\theta_i), \beta_{-i}^k(\theta_{-i}))u_i(s, w_i, \beta_i^k(\theta_i)) \\ -\varphi_i(\beta_i(\theta_i), \beta_{-i}(\theta_{-i}))u_i(s, w_i, \beta_i(\theta_i)) \end{array} \right] f(\theta_{-i}|\theta_i)d\theta_{-i} + \Delta/3.
\end{aligned}
$$

On the set where convergence is uniform, $\Theta_{-i} \setminus \Theta_{-i}^\varepsilon$, for k sufficiently large $\beta^k = \beta$ since the range of β^k is a finite set. Thus, for k sufficiently large, $U_i(\beta_i^k(\theta_i), \beta_{-i}^k|\theta_i) - U_i(\beta_i(\theta_i), \beta_{-i}|\theta_i) < 0 + \Delta/3$. By a similar argument, $U_i(\hat{\beta}_i(\theta_i), \beta_{-i}|\theta_i) - U_i(\hat{\beta}_i(\theta_i), \beta_{-i}^k|\theta_i) < \Delta/3$ for k sufficiently large. But, this implies that for k sufficiently large,

$$
\begin{aligned}
\Delta &= \left[U_i(\hat{\beta}_i, \beta_{-i}) - U_i(\hat{\beta}_i, \beta_{-i}^k)\right] + \left[U_i(\hat{\beta}_i, \beta_{-i}^k) - U_i(\beta_i^k, \beta_{-i}^k)\right] + \left[U_i(\beta_i^k, \beta_{-i}^k) - U_i(\beta_i, \beta_{-i})\right] \\
&< \Delta/3 + \underbrace{U_i(\hat{\beta}_i, \beta_{-i}^k) - U_i(\beta_i^k, \beta_{-i}^k)}_{\leq 0} + \Delta/3 \leq 2\Delta/3,
\end{aligned}
$$

which is a contradiction. (The middle term is nonpositive since β_i^k is a best response bid.) $\square$

To prove Theorem 3(b) we consider a sequence of discretized auctions with increasingly finer sets of admissible bids. Athey (2001) and Reny and Zamir (2004), among others, also consider such sequences. Specifically suppose that in auction k, bidder i's set of feasible bids is $\mathcal{B}_i^k = \{\ell_i\} \cup \left([r_i, \bar{B}] \cap \mathcal{P}_i^k\right)$ where $\mathcal{P}_i^k = \{-\varepsilon_i + m/2^k \mid m = 1, 2, \ldots\}$. $\{\varepsilon_i\}$ are small fixed constants ensuring that $\mathcal{P}_i^k \cap \mathcal{P}_j^k = \varnothing$ for all i, j, k. Contemporaneously, take $\tilde{\epsilon}^k \to 0$ as well. Let β^k be a corresponding sequence of equilibria.

Lemma C.6 (Theorem 3(b))**.** *Suppose bidders have private values, $\mathcal{B}_i = \{\ell_i\} \cup [r_i, \infty)$, and ties among high bidders are resolved with a uniform randomization. There exists an equilibrium in $\geq_i$-nondecreasing strategies.*

Proof. This is a consequence of better-reply security (Reny, 1999); see especially Reny (2011, Corollary 5.2), whom we follow, and also Jackson and Swinkels (2005). Briefly, for k very large β^k is an ϵ-equilibrium of the first-price auction where all bidders can place bids from $\mathcal{B}_i$. Given any $\geq_i$-nondecreasing strategy profile β_{-i}, bidder i can approximate the payoff he would receive from the (feasible) strategy $\beta_i \colon \Theta \to \mathcal{B}_i$. Specifically, suppose bidder i bids

$$\tilde{\beta}_i^k(\theta_i) = \begin{cases} \ell_i & w_i < r_i + \frac{1}{k} \\ \max\left\{b \in \mathcal{B}_i^k \mid b < \min\{\beta_i(\theta_i) + \frac{1}{2^k}, w_i\}\right\} & w_i \geq r_i + \frac{1}{k} \end{cases}.$$

As $k \to \infty$, the ex ante loss to bidder i from using $\tilde{\beta}_i^k$ versus β_i becomes arbitrarily small and this conclusion holds versus all strategies that the other bidders may follow. $\square$

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First-Price Auctions with Budget Constraints

Online Appendix

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A Examples

This section presents calculations pertaining to Examples 1, 3, and 4 from the main text.

A.1 Example 1

Noting symmetry, we henceforth suppress bidder subscripts. Let $H\colon [0,\overline{s}] \to [0,1]$ be the cumulative distribution function (c.d.f.) of value signals and $h(s)$ its density. With probability $p \in (0,1)$, a bidder's budget is $\underline{w}$ and $0 < \underline{w} < \overline{s} - \int_0^{\overline{s}} H(z)dz$. With probability $1-p$, it is $\overline{w} > \overline{s} - \int_0^{\overline{s}} H(z)dz$.

Remark A.1. Example 1 corresponds to the following parameters: $\overline{s} = 1$, $H(s) = s$, $\underline{w} = 1/4$, $\overline{w} = 3/4$, and $p = 1/2$.

Remark A.2. Define $s' \in (\underline{w}, \overline{s})$ as the unique value such that $\underline{w} = s' - \int_0^{s'} (H(z)/H(s'))dz$.

Lemmas A.1 and A.2 define the values $\tilde{s}$ and $\tilde{s}'$. These are the points of discontinuity in the bidding strategy introduced in (A.1) below.

Lemma A.1. *There exists a unique $\tilde{s} \in (\underline{w}, s')$ such that $\int_0^{\tilde{s}} H(z)dz = [p+(1-p)H(\tilde{s})](\tilde{s}-\underline{w})$.*

Proof. Let $\tau(s) := [p + (1-p)H(s)](s - \underline{w}) - \int_0^s H(z)dz$. It suffices to show that $\tau(s)$ crosses zero exactly once. When $s = \underline{w}$, $\tau(\underline{w}) < 0$. If instead $s = s'$, $\tau(s') = [p + (1-p)H(s')](s' -$

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$\underline{w}) - H(s')(s' - \underline{w}) > 0$. Thus, there exists $\tilde{s} \in (\underline{w}, s')$ such that $\tau(\tilde{s}) = 0$. Moreover, $\tau'(s) = p(1 - H(s)) + (1 - p)(s - \underline{w})h(s) > 0$ for all $\underline{w} < s < \overline{s}$. Thus, $\tau(s)$ is strictly increasing and $\tilde{s}$ is unique. $\square$

Lemma A.2. *Define $\tilde{s}$ as in Lemma A.1. There exists a unique $\tilde{s}' \in (\tilde{s}, \overline{s})$ such that*

$$\int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{\tilde{s}'} [pH(z) + (1-p)H(\tilde{s})]dz = \left[(1-p)H(\tilde{s}) + p\frac{1 + H(\tilde{s}')}{2}\right](\tilde{s}' - \underline{w}).$$

Proof. Let

$$\begin{aligned} \tau(s) &:= \int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{s} [(1-p)H(\tilde{s}) + pH(z)]dz, \text{ and} \\ \chi(s) &:= \left[(1-p)H(\tilde{s}) + p\frac{1+H(s)}{2}\right](s - \underline{w}). \end{aligned}$$

At $s = \tilde{s}$,

$$\begin{aligned} \tau(\tilde{s}) &= \int_0^{\tilde{s}} H(z)dz \\ &= [p + (1-p)H(\tilde{s})](\tilde{s} - \underline{w}) \\ &> \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s})}{2}\right](\tilde{s} - \underline{w}) = \chi(\tilde{s}). \end{aligned}$$

Instead, if $s = \overline{s}$,

$$\begin{aligned} \tau(\overline{s}) &= [p + (1-p)H(\tilde{s})](\tilde{s} - \underline{w}) + \int_{\tilde{s}}^{\overline{s}} [(1-p)H(\tilde{s}) + pH(z)]dz \\ &< [p + (1-p)H(\tilde{s})](\tilde{s} - \underline{w}) + [p + (1-p)H(\tilde{s})](\overline{s} - \tilde{s}) \\ &= [p + (1-p)H(\tilde{s})](\overline{s} - \underline{w}) = \chi(\overline{s}). \end{aligned}$$

Thus, there exists $\tilde{s}' \in (\tilde{s}, \overline{s})$ such that $\tau(\tilde{s}') = \chi(\tilde{s}')$. To verify uniqueness, note that $\tau'(s) = (1-p)H(\tilde{s}) + pH(s)$ and $\chi'(s) = (1-p)H(\tilde{s}) + p(1 + H(s))/2 + psh(s)/2$. Hence, $\tau'(s) < \chi'(s)$ and $s \mapsto \tau(s) - \chi(s)$ is strictly decreasing. Thus, $\tilde{s}'$ is unique. $\square$

Proposition A.1. *Define $\tilde{s}$ and $\tilde{s}'$ as in Lemmas A.1 and A.2. The strategy profile where* *all bidders follow the strategy*

$$\beta(s,w) = \begin{cases} s - \dfrac{1}{H(s)} \displaystyle\int_0^s H(z)dz & \textit{if } s \in [0,\tilde{s}] \ \& \ w \in \{\underline{w}, \overline{w}\} \\ s - \dfrac{\int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{s} [p + (1-p)H(z)]dz}{p + (1-p)H(s)} & \textit{if } s \in (\tilde{s}, \overline{s}] \ \& \ w = \overline{w} \\ s - \dfrac{\int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{s} [pH(z) + (1-p)H(\tilde{s})]dz}{pH(s) + (1-p)H(\tilde{s})} & \textit{if } s \in (\tilde{s}, \tilde{s}'] \ \& \ w = \underline{w} \\ \underline{w} & \textit{if } s \in (\tilde{s}', \overline{s}] \ \& \ w = \underline{w} \end{cases} \tag{A.1}$$

is a symmetric Bayesian-Nash equilibrium.

Proof. The proof mirrors analogous arguments in the absence of budget constraints (for example, see Krishna, 2002, pp. 17–18). To simplify notation, let $\bar{\beta}(s) := \beta(s, \overline{w})$ and $\underline{\beta}(s) := \beta(s, \underline{w})$. Let $U(b|s,w)$ be the expected utility of bidder i of type (s,w) when he bids b and the other bidder follows $\beta(s,w)$. We proceed to rule out deviations by each type of bidder. It is sufficient to rule out deviations to other bids in the range of $\beta(s,w)$.

Case 1. Consider bidder i of type $s \in [0, \tilde{s}]$ with budget $\overline{w}$. Given (A.1), $U(\bar{\beta}(s)|s,\overline{w}) = \int_0^s H(z)dz$. There are four kinds of alternative bids to consider.

(a) A deviation to $\bar{\beta}(x)$, $x \in [0, \tilde{s}]$, is unprofitable. The standard argument confirming the equilibrium in a symmetric, first-price, sealed-bid auction applies.

(b) A deviation to $\underline{\beta}(x)$, $x \in (\tilde{s}, \tilde{s}']$, yields a payoff of $U(\underline{\beta}(x)|s,\overline{w}) = [(1-p)H(\tilde{s}) + pH(x)](s-x) + \int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{x} [(1-p)H(\tilde{s}) + pH(z)]dz$. The net gain is

$$\begin{aligned} & U(\underline{\beta}(x)|s,\overline{w}) - U(\bar{\beta}(s)|s,\overline{w}) \\ &= [(1-p)H(\tilde{s}) + pH(x)](s-x) + \int_s^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{x} [(1-p)H(\tilde{s}) + pH(z)]dz \\ &= \int_s^{\tilde{s}} H(z)dz + [(1-p)H(\tilde{s}) + pH(x)](s-\tilde{s}) + pH(x)(\tilde{s}-x) + \int_{\tilde{s}}^{x} pH(z)dz \\ &\leq H(\tilde{s})(\tilde{s}-s) + [(1-p)H(\tilde{s}) + pH(\tilde{s})](s-\tilde{s}) + pH(x)(\tilde{s}-x) + pH(x)(x-\tilde{s}) = 0. \end{aligned}$$

Thus, $U(\underline{\beta}(x)|s,\overline{w}) \leq U(\bar{\beta}(s)|s,\overline{w})$.

(c) Given β, the bid $\underline{w}$ is placed with positive probability by bidder j. If $s \leq \overline{w}$, bidding $\overline{w}$ cannot benefit bidder i. Else, if $s > \overline{w}$, the bid $\underline{w} + \epsilon$ gives bidder i a strictly higher payoff than $\overline{w}$. Thus, sub-case (d) immediately below applies.

(d) Consider a deviation to a bid $\bar{\beta}(x)$, $x \in (\tilde{s}, \overline{s}]$. Bidder i's expected payoff is $U(\bar{\beta}(x)|s,w) = [p + (1-p)H(x)](s-x) + \int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{x}[p + (1-p)H(z)]dz$. The net gain is

$$
\begin{aligned}
&U(\bar{\beta}(x)|s,\overline{w}) - U(\bar{\beta}(s)|s,\overline{w}) \\
&= \int_s^{\tilde{s}} H(z)dz + [p + (1-p)H(x)](s-\tilde{s}) \\
&\qquad + [p + (1-p)H(x)](\tilde{s}-x) + \int_{\tilde{s}}^{x} [p + (1-p)H(z)]dz \\
&\leq H(\tilde{s})(\tilde{s}-s) + [p + (1-p)H(x)](s-\tilde{s}) + \int_{\tilde{s}}^{x} (1-p)[H(z) - H(x)]dz \leq 0.
\end{aligned}
$$

The final inequality follows from $x \geq \tilde{s} \geq s$. Thus, $U(\bar{\beta}(x)|s,\overline{w}) \leq U(\bar{\beta}(s)|s,\overline{w})$.

Case 2. Consider bidder i of type $s \in (\tilde{s}, \overline{s}]$ with budget $w = \overline{w}$. If this bidder follows the prescribed strategy, $U(\bar{\beta}(s)|s,\overline{w}) = \int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{s}[p + (1-p)H(z)]dz$. There are four kinds of alternative bids to consider.

(a) A deviation to $\bar{\beta}(x)$, $x \in [0, \tilde{s}]$, yields a payoff of $U(\bar{\beta}(x)|s,\overline{w}) = H(x)(s-x) + \int_0^{x} H(z)dz$. The net gain is

$$
\begin{aligned}
&U(\bar{\beta}(x)|s,\overline{w}) - U(\bar{\beta}(s)|s,\overline{w}) \\
&= H(x)(s-x) + \int_0^{x} H(z)dz - \int_0^{\tilde{s}} H(z)dz - \int_{\tilde{s}}^{s} [p + (1-p)H(z)]dz \\
&= H(x)(\tilde{s}-x) + \int_{\tilde{s}}^{x} H(z)dz + H(x)(s-\tilde{s}) - \int_{\tilde{s}}^{s} [p + (1-p)H(z)]dz \\
&\leq \int_x^{\tilde{s}} [H(x) - H(z)]dz + H(x)(s-\tilde{s}) - [p + (1-p)H(\tilde{s})](s-\tilde{s}) \leq 0.
\end{aligned}
$$

Therefore, this deviation is not profitable.

(b) A deviation to $\underline{\beta}(x)$, $x \in [\tilde{s}, \tilde{s}']$, yields a payoff of $U(\underline{\beta}(x)|s,\overline{w}) = [(1-p)H(\tilde{s}) +$

$pH(x)](s-x)+\int_0^{\tilde{s}} H(z)dz+\int_{\tilde{s}}^{x}[(1-p)H(\tilde{s})+pH(z)]dz$. The net gain is

$$\begin{aligned}
&U(\underline{\beta}(x)|s,\overline{w})-U(\bar{\beta}(s)|s,\overline{w})\\
&=[(1-p)H(\tilde{s})+pH(x)](s-x)+\int_{\tilde{s}}^{x}[(1-p)H(\tilde{s})+pH(z)]dz\\
&\qquad-\int_{\tilde{s}}^{s}[p+(1-p)H(z)]dz\\
&=[(1-p)H(\tilde{s})+pH(x)](\tilde{s}-x)+\int_{\tilde{s}}^{x}[(1-p)H(\tilde{s})+pH(z)]dz\\
&\qquad+[(1-p)H(\tilde{s})+pH(x)](s-\tilde{s})-\int_{\tilde{s}}^{s}[p+(1-p)H(z)]dz\\
&\leq\int_{\tilde{s}}^{x}p[H(z)-H(x)]dz\\
&\qquad+[(1-p)H(\tilde{s})+pH(x)](s-\tilde{s})-[p+(1-p)H(\tilde{s})](s-\tilde{s})\\
&=\int_{\tilde{s}}^{x}p[H(z)-H(x)]dz+p(s-\tilde{s})(H(x)-1)\leq 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

(c) Since $\underline{w}<\tilde{s}<s$, the bidder can strictly improve upon the bid $\underline{w}$ by bidding ϵ more. Thus, the conclusion from sub-case (d) immediately below applies.

(d) A deviation to $\bar{\beta}(x)$, $x\in(\tilde{s},\overline{s}]$, yields a payoff of $U(\bar{\beta}(x)|s,\overline{w})=[p+(1-p)H(x)](s-x)+\int_0^{\tilde{s}} H(z)dz+\int_{\tilde{s}}^{x}[p+(1-p)H(z)]dz$. The net gain is

$$\begin{aligned}
U(\bar{\beta}(x)|s,\overline{w})-U(\bar{\beta}(s)|s,\overline{w})&=[p+(1-p)H(x)](s-x)+\int_{s}^{x}[p+(1-p)H(z)]dz\\
&=\int_{s}^{x}(1-p)[H(z)-H(x)]dz\leq 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

Case 3. Consider bidder i of type $s\in[0,\tilde{s}]$ with budget $w=\underline{w}$. By identical arguments to those presented above, this bidder will not have a profitable deviation to any bid, except possibly $\underline{w}$. This deviation yields a payoff of $U(\underline{w}|s,\underline{w})=$

$[(1-p)H(\tilde{s}) + p(1+H(\tilde{s}'))/2]\,(s-\underline{w})$. The net gain is

$$\begin{aligned}
&U(\underline{w}|s,\underline{w}) - U(\underline{\beta}(s)|s,\underline{w}) \\
&= \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s}')}{2}\right](s-\tilde{s}') \\
&\qquad + \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s}')}{2}\right](\tilde{s}'-\underline{w}) - \int_0^s H(z)dz \\
&= \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s}')}{2}\right](s-\tilde{s}') + \int_s^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{\tilde{s}'} [(1-p)H(\tilde{s}) + pH(z)]dz \\
&= \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s}')}{2}\right](s-\tilde{s}) + \int_s^{\tilde{s}} H(z)dz \\
&\qquad + \left[(1-p)H(\tilde{s}) + p\frac{1+H(\tilde{s}')}{2}\right](\tilde{s}-\tilde{s}') + \int_{\tilde{s}}^{\tilde{s}'} [(1-p)H(\tilde{s}) + pH(z)]dz \le 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

Case 4. Consider bidder i of type $s \in (\tilde{s}, \tilde{s}']$ with budget $w = \underline{w}$. If this bidder follows the prescribed strategy, $U(\underline{\beta}(s)|s,\underline{w}) = \int_0^{\tilde{s}} H(z)dz + \int_{\tilde{s}}^{s} [pH(z) + (1-p)H(\tilde{s})]dz$.

(a) A deviation to $\underline{\beta}(x)$, $x \in [0, \tilde{s}]$, yields a payoff of $U(\underline{\beta}(x)|s,\underline{w}) = H(x)(s-x) + \int_0^x H(z)dz$. The net gain is

$$\begin{aligned}
&U(\underline{\beta}(x)|s,\underline{w}) - U(\underline{\beta}(s)|s,\underline{w}) \\
&= H(x)(s-x) + \int_0^x H(z)dz - \int_0^{\tilde{s}} H(z)dz - \int_{\tilde{s}}^s [(1-p)H(\tilde{s}) + pH(z)]dz \\
&= H(x)(\tilde{s}-x) + \int_{\tilde{s}}^x H(z)dz + H(x)(s-\tilde{s}) - \int_{\tilde{s}}^s [(1-p)H(\tilde{s}) + pH(z)]dz \\
&\le 0 + H(x)(s-\tilde{s}) - [(1-p)H(\tilde{s}) + pH(\tilde{s})](s-\tilde{s}) \\
&= [H(x) - H(\tilde{s})](s-\tilde{s}) \le 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

(b) A deviation to $\underline{\beta}(x)$, $x \in [\tilde{s}, \tilde{s}']$, yields a payoff of $U(\underline{\beta}(x)|s,\underline{w}) = [(1-p)H(\tilde{s}) +$

$pH(x)](s-x)+\int_0^{\tilde{s}} H(z)dz+\int_{\tilde{s}}^{x}[(1-p)H(\tilde{s})+pH(z)]dz$. The net gain is

$$\begin{aligned}
&U(\underline{\beta}(x)|s,\underline{w})-U(\underline{\beta}(s)|s,\underline{w})\\
&=[(1-p)H(\tilde{s})+pH(x)](s-x)+\int_s^x[(1-p)H(\tilde{s})+pH(z)]dz\\
&=\int_s^x p[H(z)-H(x)]dz\leq 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

(c) A deviation to $\underline{w}$ can be ruled out by an argument similar to that from case 3.

Case 5. Consider bidder i of type $s\in(\tilde{s}',\overline{s}]$ with budget $w=\underline{w}$. If this bidder follows the prescribed strategy, $U(\underline{\beta}(s)|s,\underline{w})=[(1-p)H(\tilde{s})+p(1+H(\tilde{s}'))/2]\,(s-\underline{w})$.

(a) A deviation to $\underline{\beta}(x)$, $x\in[0,\tilde{s}]$, yields a payoff of $U(\underline{\beta}(x)|s,\underline{w})=H(x)(s-x)+\int_0^x H(z)dz$. The net gain is

$$\begin{aligned}
&U(\underline{\beta}(x)|s,\underline{w})-U(\underline{\beta}(s)|s,\underline{w})\\
&=H(x)(s-x)+\int_0^x H(z)dz-\left[(1-p)H(\tilde{s})+p\frac{1+H(\tilde{s}')}{2}\right](s-\underline{w})\\
&=H(x)(s-x)+\int_0^x H(z)dz-\int_0^{\tilde{s}} H(z)dz-\int_{\tilde{s}}^{\tilde{s}'}[(1-p)H(\tilde{s})+pH(z)]dz\\
&\quad-\left[(1-p)H(\tilde{s})+p\frac{1+H(\tilde{s}')}{2}\right](s-\tilde{s}')\\
&=H(x)(\tilde{s}-x)+\int_{\tilde{s}}^x H(z)dz+H(x)(\tilde{s}'-\tilde{s})-\int_{\tilde{s}}^{\tilde{s}'}[(1-p)H(\tilde{s})+pH(z)]dz\\
&\quad+H(x)(s-\tilde{s}')-\left[(1-p)H(\tilde{s})+p\frac{1+H(\tilde{s}')}{2}\right](s-\tilde{s}')\leq 0.
\end{aligned}$$

Therefore, this deviation is not profitable.

(b) A deviation to $\underline{\beta}(x)$, $x\in[\tilde{s},\tilde{s}']$, can be ruled out with reasoning similar to the preceding case.

The preceding cases have ruled out all possible deviations, thus completing the proof. $\square$

A.2 Example 3

There are two ex ante symmetric bidders. Each bidder's valuation is $v(s_i, s_j) = s_i + s_j$. Each bidder's value-signal is independently distributed according to the c.d.f. $H(s) = \sqrt{s}$ on $[0, 1]$. Ties are resolved with a uniform randomization. In the absence of budget constraints, $\alpha(s) = 2s/3$ is the equilibrium bidding strategy (Milgrom and Weber, 1982).

Proposition A.2. *Consider the auction described above. Additionally, suppose both bidders face a common budget constraint of 1/3.*

(a) There exists an equilibrium where each bidder adopts the bidding strategy

$$\beta_A(s) = \begin{cases} 2s/3 & \text{if } s \leq 1/9 \\ 1/3 & \text{if } s > 1/9 \end{cases}.$$

(b) There exists an equilibrium where each bidder adopts the bidding strategy $\beta_B(s) = 1/3$.

Proof. (a) Suppose bidder j adopts the strategy $\beta_A(s)$. Clearly, all bids in the range $(2/27, 1/3)$ are dominated by the bid $\beta_A(1/9) = 2/27$. Thus, to confirm that β_A is an equilibrium it is sufficient to rule out potential deviations to other bids in the range of β_A.

Consider bidder i with value-signal $s \leq 1/9$. If he bids $\beta_A(s) = 2s/3$, his expected payoff is $(2/3)s^{3/2}$. Since 2s/3 is the symmetric equilibrium strategy in the absence of budget constraints, bidder i has no profitable deviation to any bid in the range $[0, 2/27]$. If he bids 1/3, his expected payoff is

$$\sqrt{\frac{1}{9}}\left(s + \mathbb{E}\left[S_j | S_j \leq \frac{1}{9}\right] - \frac{1}{3}\right) + \frac{1 - \sqrt{1/9}}{2}\left(s + \mathbb{E}\left[S_j | S_j > \frac{1}{9}\right] - \frac{1}{3}\right) = \frac{2}{9}\left(3s - \frac{2}{9}\right).$$

For all $s \leq 1/9$, $(2/9)(3s - 2/9) \leq (2/3)s^{3/2}$. Thus, $\beta_A(s)$ is an optimal bid for bidder i when $s \leq 1/9$.

Now consider bidder i with value signal $s > 1/9$. If he bids $\beta_A(x) = 2x/3$, for some $x \leq 1/9$, his expected payoff is $\sqrt{x}\,(s + \mathbb{E}[S_j | S_j \leq x] - 2x/3) = \sqrt{x}(s - x/3) \leq (1/3)(s - 1/27)$. Clearly, $(1/3)(s - 1/27) \leq (2/9)(3s - 2/9)$ for all $s \geq 1/9$. Thus, he has no profitable deviation from 1/3 to any bid $b \in [0, 2/27]$.

As the setting is symmetric, we conclude that β_A is a symmetric equilibrium strategy.

(b) Suppose that bidder j adopts the strategy $\beta_B(s) = 1/3$. The expected payoff of bidder i with value-signal s when he bids $1/3$ is

$$\frac{1}{2}\left(s+\mathbb{E}[S_j]-\frac{1}{3}\right)=\frac{1}{2}\left(s+\frac{1}{3}-\frac{1}{3}\right)=\frac{s}{2}.$$

If bidder i bids less than $1/3$, his expected payoff is zero as he loses the auction with certainty. Furthermore, bids above $1/3$ are not feasible. Thus, $1/3$ is a best response for bidder i given bidder j's strategy. And so, β_B is a symmetric equilibrium strategy. $\square$

A.3 Example 4

There are two bidders. Value-signals are distributed uniformly on the unit interval. A bidder's budget is $\underline{w}=1/4$ with probability $1/2$; it is $\overline{w}=3/4$ with probability $1/2$. If a bidder of type (s,w) wins the auction with the bid b, his payoff is $(s-b)^{w+1/4}$; else it is zero.

Proposition A.3. *The following is a symmetric equilibrium strategy in Example 4.*

$$\beta(s,3/4)=\begin{cases}\dfrac{s}{2} & \text{if } s\in[0,\tilde{s}]\\ \dfrac{2s^2+13\sqrt{11}-42}{4(s+1)} & \text{if } s\in(\tilde{s},1]\end{cases}$$

$$\beta(s,1/4)=\begin{cases}\dfrac{2s}{3} & \text{if } s\in[0,\tilde{s}'']\\ \dfrac{16s^3+24(\sqrt{11}-3)s^2+33(19\sqrt{11}-63)}{24(s+\sqrt{11}-3)^2} & \text{if } s\in(\tilde{s}'',\tilde{s}']\\ \dfrac{1}{4} & \text{if } s\in(\tilde{s}',1]\end{cases}$$

where $\tilde{s}=\sqrt{11}-3\approx 0.317$, $\tilde{s}''=3(\sqrt{11}-3)/4\approx 0.237$, *and* $\tilde{s}'\approx 0.298$ *solves the equation*

$$\left(\frac{\tilde{s}'+\tilde{s}}{2}\right)(\tilde{s}-q(\tilde{s}))^{\frac{1}{2}}=\left(\frac{1}{2}\cdot\frac{\tilde{s}'+1}{2}+\frac{1}{2}\tilde{s}\right)(\tilde{s}-1/4)^{\frac{1}{2}}$$

where

$$q(s)=\frac{16s^3+24(\sqrt{11}-3)s^2+33(19\sqrt{11}-63)}{24(s+\sqrt{11}-3)^2}.$$

Proof. We mimic the proof of Proposition A.1. Let $\bar{\beta}(s):=\beta(s,3/4)$ and $\underline{\beta}(s):=\beta(s,1/4)$.

Case 1. Consider bidder i with budget $w=3/4$ and value-signal $s\in[0,\tilde{s}]$. The bid $\bar{\beta}(x)$, $x\in[0,\tilde{s}]$, defeats a high-budget opponent with a value-signal less than x and a low-budget opponent with a value-signal less than $3x/4$. The expected payoff is

$U(\bar{\beta}(x)|s,\overline{w}) = (7x/8)(s - x/2)$. Maximizing this expression with respect to x, subject to the constraint $x \leq \tilde{s}$, gives a solution of $x = s$. Thus, $\bar{\beta}(s)$ is the optimal bid when constrained to a bid in the range $[0, \tilde{s}/2]$.

Instead, suppose i bids $\bar{\beta}(x)$, $x \in (\tilde{s}, 1]$. This bid defeats a low-budget opponent and a high-budget opponent with a value-signal less than x. The expected payoff is $U(\bar{\beta}(x)|s,\overline{w}) = ((1+x)/2)(s-\bar{\beta}(x))$. Noting that $dU(\bar{\beta}(x)|s,\overline{w})/dx = (s-x)/2 \leq 0$ and given the definition of $\tilde{s}$, we see that $U(\bar{\beta}(x)|s,\overline{w}) \leq \lim_{x\to\tilde{s}^+} U(\bar{\beta}(x)|s,\overline{w}) = ((1+\tilde{s})/2)\,(s-1/4) = (\sqrt{11}-2)(4s-1)/8 \leq 7s^2/16 = U(\bar{\beta}(s)|s,\overline{w})$. Hence, this proposed deviation is not profitable.

The remaining candidate deviation is to the bid $\underline{\beta}(x)$, $x \in [\tilde{s}'', \tilde{s}')$. For $x \in [\tilde{s}'', \tilde{s}')$, let

$$\Delta(x,s) := U(\bar{\beta}(s)|s,\overline{w}) - U(\underline{\beta}(x)|s,\overline{w}) = \frac{7s^2}{16} - \left(\frac{x+\tilde{s}}{2}\right)(s - \underline{\beta}(x)).$$

It is sufficient to establish that $\Delta(x,s) \geq 0$ for all $x \in [\tilde{s}'', \tilde{s}')$ and $s \leq \tilde{s}$. From the definition of $\tilde{s}$ and $\tilde{s}''$ (noting that $\underline{\beta}(s)$ is continuous at $\tilde{s}''$), $\Delta(\tilde{s}'', \tilde{s}) = 0$. Next, note that $\partial\Delta(\tilde{s}'', s)/\partial s = 7(s-\tilde{s})/8 \leq 0$. Hence, for all $s \leq \tilde{s}$, $\Delta(\tilde{s}'', s) \geq 0$. Thus, it is sufficient to confirm that $\partial\Delta(x,s)/\partial x \geq 0$ for all $s \in [0, \tilde{s}]$. A direct computation gives

$$\begin{aligned}\frac{\partial\Delta(x,s)}{\partial x} &= \frac{1}{48}\left(8(\sqrt{11}-3-3s) + 32x - \frac{49(19\sqrt{11}-63)}{(x+\sqrt{11}-3)^2}\right) \\ &\geq \frac{1}{48}\left(8(\sqrt{11}-3-3\tilde{s}) + 32\tilde{s}'' - \frac{49(19\sqrt{11}-63)}{(\tilde{s}''+\sqrt{11}-3)^2}\right) = 0.\end{aligned}$$

Therefore, a high-budget bidder of type $s \leq \tilde{s}$ has no incentive to deviate from $\bar{\beta}(s)$.

Case 2. Consider bidder i with budget $w = 3/4$ and value-signal $s \in (\tilde{s}, 1]$. By an argument analogous to the preceding case, all deviations to bids $\bar{\beta}(x)$, $x \in [0,1]$, can be ruled out. We therefore consider a possible deviation to the bid $\underline{\beta}(x)$, $x \in [\tilde{s}'', \tilde{s}')$. Let

$$\Delta(x,s) := U(\bar{\beta}(s)|s,\overline{w}) - U(\underline{\beta}(x)|s,\overline{w}) = \left(\frac{1+s}{2}\right)(s - \bar{\beta}(s)) - \left(\frac{x+\tilde{s}}{2}\right)(s - \underline{\beta}(x)).$$

It is sufficient to verify that $\Delta(x,s) \geq 0$ for $s \in (\tilde{s}, 1]$ and $x \in [\tilde{s}'', \tilde{s}']$. As before,

$\Delta(\tilde{s}'', \tilde{s}) = 0$. And for any $x \in [\tilde{s}'', \tilde{s}']$,

$$\frac{\partial \Delta(x,s)}{\partial s} = \frac{4 - \sqrt{11} + s - x}{2} \geq \frac{4 - \sqrt{11} + \tilde{s} - x}{2} \geq 0.$$

Thus, it is sufficient to confirm that $\partial \Delta(x, \tilde{s})/\partial x \geq 0$. A direct calculation gives

$$\begin{aligned}\frac{\partial \Delta(x,\tilde{s})}{\partial x} &= \frac{2x}{3} - \frac{49(19\sqrt{11} - 63)}{48(x + \sqrt{11} - 3)^2} - \frac{\sqrt{11}}{3} + 1 \\ &\geq \frac{2\tilde{s}''}{3} - \frac{49(19\sqrt{11} - 63)}{48(\tilde{s}'' + \sqrt{11} - 3)^2} - \frac{\sqrt{11}}{3} + 1 = 0.\end{aligned}$$

Thus, a high-budget bidder with a value-signal of $s > \tilde{s}$ has no profitable deviation from $\bar{\beta}(s)$.

Case 3. Consider bidder i with budget $w = 1/4$ and value-signal $s \in [0, \tilde{s}'']$. The bid $\underline{\beta}(x)$, $x \in [0, \tilde{s}'']$, defeats a low-budget opponent with a value-signal less than x and a high-budget opponent with a value-signal less than $4x/3$. The expected payoff is

$$U(\underline{\beta}(x)|s, \underline{w}) = \frac{7x}{6}\left(s - \frac{2x}{3}\right)^{1/2}.$$

Maximizing the above expression with respect to x (subject to $x \leq \tilde{s}''$) gives a solution of $x = s$. Thus, $\underline{\beta}(s)$ is the best bid among bids in the range $[0, 2\tilde{s}''/3]$.

If instead bidder i bids $\underline{\beta}(x)$, $x \in [\tilde{s}'', \tilde{s}']$, his expected payoff is

$$U(\underline{\beta}(x)|s, \underline{w}) = \left(\frac{x + \tilde{s}}{2}\right)(s - \underline{\beta}(x))^{\frac{1}{2}}.$$

Substituting and differentiating with respect to x gives

$$\frac{d}{dx}U(\underline{\beta}(x)|s, \underline{w}) = \frac{\sqrt{6}(s - x)}{\sqrt{8(3s + \sqrt{11} - 3) - 16x - \frac{49(19\sqrt{11}-63)}{(x+\sqrt{11}-3)^2}}} \leq 0.$$

This implies $U(\underline{\beta}(x)|s, \underline{w}) \leq U(\underline{\beta}(\tilde{s}'')|s, \underline{w}) \leq U(\underline{\beta}(s)|s, \underline{w})$.

Finally, since $\tilde{s}'' = 3(\sqrt{11} - 3)/4 \approx 0.23$, the bid $\underline{w} = 1/4$ would yield a negative payoff. Therefore, this bidder does not have a profitable deviation.

Case 4. Consider bidder i with budget $w = 1/4$ and value-signal $s \in (\tilde{s}'', \tilde{s}']$. By an argument analogous to the preceding case, a profitable deviation to the bid $\underline{\beta}(x)$, $x \in [0, \tilde{s}']$, can be ruled out. We therefore consider only the bid of $\underline{w} = 1/4$. Let $\Delta(s) := U(\underline{\beta}(s)|s, \underline{w}) - U(\underline{w}|s, \underline{w})$. From the definition of $\tilde{s}'$, $\Delta(\tilde{s}') = 0$. Therefore, it is sufficient to verify that $\Delta'(s) \leq 0$; equivalently, $dU(\underline{\beta}(s)|s, \underline{w})/ds \leq dU(\underline{w}|s, \underline{w})/ds$. This latter fact is true since

$$\frac{d}{ds}U(\underline{w}|s, \underline{w}) = \frac{\tilde{s}' + 2\sqrt{11} - 5}{4\sqrt{4s - 1}} \geq \frac{\tilde{s}'' + 2\sqrt{11} - 5}{4\sqrt{4\tilde{s} - 1}} \approx 0.901$$

and

$$\begin{aligned}\frac{d}{ds}U(\underline{\beta}(s)|s, \underline{w}) &= \frac{\sqrt{\frac{3}{2}}\left(s + \sqrt{11} - 3\right)}{\sqrt{8s - \frac{49\left(19\sqrt{11}-63\right)}{\left(s+\sqrt{11}-3\right)^2} + 8(\sqrt{11} - 3)}} \\ &\leq \frac{\sqrt{\frac{3}{2}}(\tilde{s} + \sqrt{11} - 3)}{\sqrt{8\tilde{s}'' - \frac{49(19\sqrt{11}-63)}{(\tilde{s}''+\sqrt{11}-3)^2} + 8(\sqrt{11} - 3)}} \approx 0.563.\end{aligned}$$

Case 5. Consider bidder i with budget $w = 1/4$ and value-signal $s \in (\tilde{s}', 1]$. By arguments analogous to those in the previous cases, we can verify that $\underline{\beta}(\tilde{s}')$ is the utility-maximizing, feasible bid other than $\underline{w}$.

To verify that $U(\underline{w}|s, \underline{w}) \geq U(\underline{\beta}(\tilde{s}')|s, \underline{w})$, recall that at $\tilde{s}'$, $U(\underline{w}|\tilde{s}', \underline{w}) \geq U(\underline{\beta}(\tilde{s}')|\tilde{s}', \underline{w})$. Thus, it is sufficient to verify that for all $s > \tilde{s}'$, $dU(\underline{\beta}(\tilde{s}')|s, \underline{w})/ds \leq dU(\underline{w}|s, \underline{w})/ds$. Observe that

$$\begin{aligned}\frac{\frac{d}{ds}U(\underline{w}|s, \underline{w})}{\frac{d}{ds}U(\underline{\beta}(\tilde{s}')|s, \underline{w})} &= \frac{(\tilde{s}' + 2\sqrt{11} - 5)\sqrt{8(3s + \sqrt{11} - 3) - 16\tilde{s}' - \frac{49(19\sqrt{11}-63)}{(\tilde{s}'+\sqrt{11}-3)^2}}}{2\sqrt{24s - 6}(\tilde{s}' + \sqrt{11} - 3)} \\ &\geq \frac{(\tilde{s}'' + 2\sqrt{11} - 5)\sqrt{8(3s + \sqrt{11} - 3) - 16\tilde{s} - \frac{49(19\sqrt{11}-63)}{(\tilde{s}''+\sqrt{11}-3)^2}}}{2\sqrt{24s - 6}(\tilde{s} + \sqrt{11} - 3)} \\ &= \frac{\sqrt{(333 + 68\sqrt{11})s - 86\sqrt{11} + \frac{502}{3}}}{8\sqrt{4s - 1}}.\end{aligned}$$

The final expression is decreasing in s. Evaluating it at $s = 1$ gives

$$\frac{\frac{d}{ds}U(\underline{w}|s, \underline{w})}{\frac{d}{ds}U(\underline{\beta}(\tilde{s}')|s, \underline{w})} \geq \frac{1}{24}\sqrt{1501 - 54\sqrt{11}} \approx 1.51 \geq 1.$$

Rearranging terms gives the desired conclusion.

As we have exhausted all possibilities, the proposed strategy is a symmetric equilibrium. $\square$

B Identification of $\bar{b}(s)$

A key step in the construction of an equilibrium strategy is the identification of $\bar{b}(s)$, the bid of an high-budget or unconstrained agent. The main text outlines its identification when the system

$$\dot{s}(s,b) = \gamma(b)(\eta(s,s) - \delta(b,s)) \qquad \dot{b}(s,b) = \lambda(s)(b - v(s,s)) \tag{B.1}$$

has one critical point.[1] Critical points occur where the nullclines, $\psi(s) \coloneqq \{b \mid \eta(s,s) = \delta(b,s)\}$ and $\nu(s) \coloneqq \{b \mid b = v(s,s)\}$, intersect. If there is one critical point, $\underline{w} = 0$, and Assumption 1 holds, then $\bar{b}(s) \equiv b^*(s)$. Recall that $b^*(\cdot)$ is a function coincident with the stable manifolds of (B.1). Below we present the identification of $\bar{b}(\cdot)$ in other cases of interest.

B.1 Multiple Critical Points and Canonical Equilibria

Consider the case in Figure B.1, which features three critical points and $\underline{w} = 0$. When there are three critical points, they alternate between saddle points and nodes. Define $b^*(s)$ as a function coincident with the stable manifolds approaching the two saddle points from below and above. In Figure B.1, this is the bold curve. This function begins in region R_1 passes through (s_0, b_0) and continues on to (s_1, b_1). On the other side of (s_1, b_1), $b^*(s)$ continues in region R_3 through (s_2, b_2) and onto region R_4. Note that $b^*(s)$ is defined for all s. As in the baseline case, we again identify $\bar{b}(s)$ with $b^*(s)$. Further analysis of the canonical equilibrium is unaffected. The construction generalizes via induction to more critical points.

B.2 Multiple Critical Points and Non-Canonical Equilibria

A similar argument to the preceding case applies to non-canonical equilibria. A particular case with three critical points is illustrated in Figure B.2. We use the function

$$\mu(s) \coloneqq \begin{cases} \min\{b^*(s), \psi(s)\} & \text{if } \psi(s) \neq \varnothing \\ \min\{b^*(s), \underline{w}\} & \text{if } \psi(s) = \varnothing \end{cases}$$

[1] Recall that $\lambda(s) \coloneqq \frac{h(s)}{1-H(s)}$, $\gamma(b) \coloneqq \frac{g(b)}{1-G(b)}$, $\eta(x,s) \coloneqq \int_x^1 \frac{v(s,y)h(y)}{1-H(x)}dy$, and $\delta(b,s) \coloneqq b + \frac{G(b)}{g(b)} + \frac{H(s)}{g(b)(1-H(s))}$.

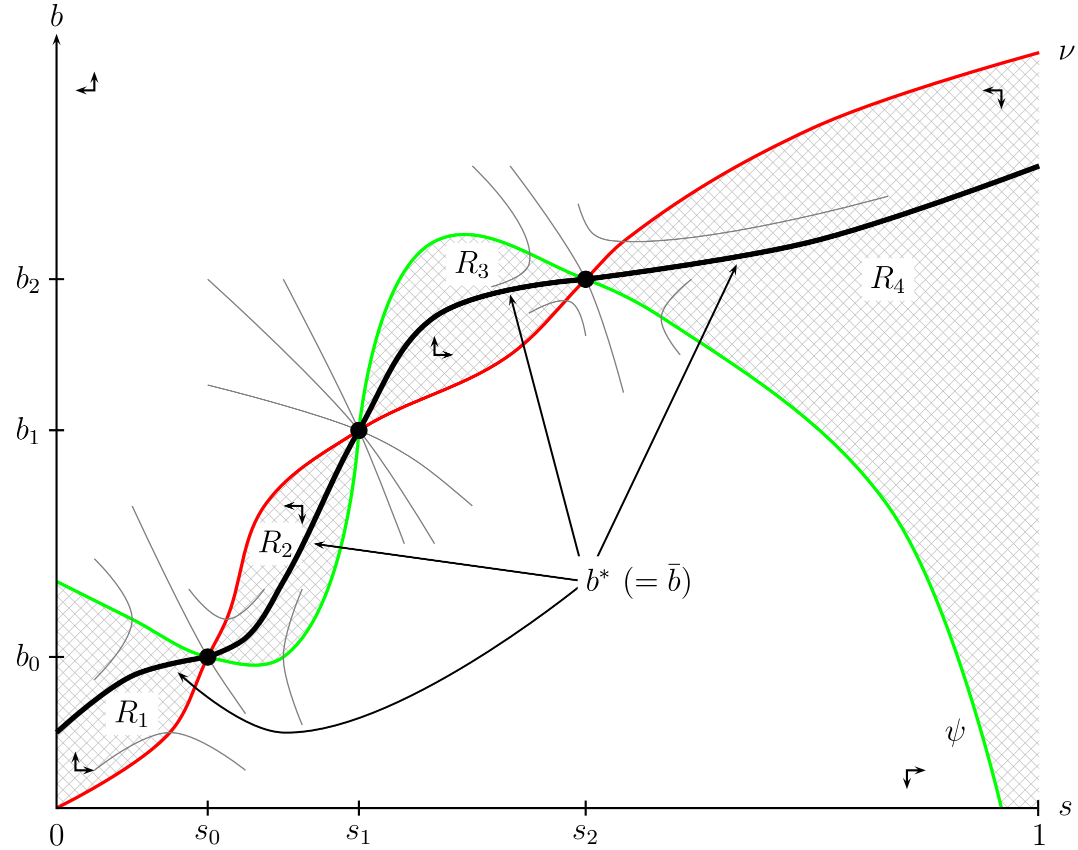


Figure B.1: Identification of $\bar{b}(s)$ when $\underline{w} = 0$. There are three critical points.

to identify the value $\tilde{s}$ at which $\bar{b}(s)$ exhibits a jump discontinuity. In Figure B.2, the function $\mu(\cdot)$ is the dashed curve defined on $[s_*, 1]$. Recall that $[s_*, 1]$ is the domain of $b^*(\cdot)$. For $s \in [s_*, s_0]$, $\mu(s) = b^*(s)$. For $s \in (s_0, s_1]$, $\mu(s) = \psi(s)$. And so on. At each point along this curve, there is an increasing solution of system (B.1) that can be extended to the boundary as an increasing function. Given $\mu(\cdot)$, the value $\tilde{s}$ solves the same indifference condition as in the baseline analysis. Given $\tilde{s}$, $\bar{b}(s)$ is defined as follows. For $s < \tilde{s}$, $\bar{b}(s) = \alpha(s)$, which is the equilibrium bidding strategy in a first-price auction in the absence of budget constraints. For $s > \tilde{s}$, $\bar{b}(s)$ follows the increasing solutions of system (B.1) starting at $(\tilde{s}, \mu(\tilde{s}))$. In the specific case of Figure B.2, $\bar{b}(s)$ follows a solution approaching the critical point (s_1, b_1) from below when $s \in (\tilde{s}, s_1)$. For $s > s_1$, $\bar{b}(s)$ continues to the boundary along $b^*(s)$ passing through (s_2, b_2) en route.

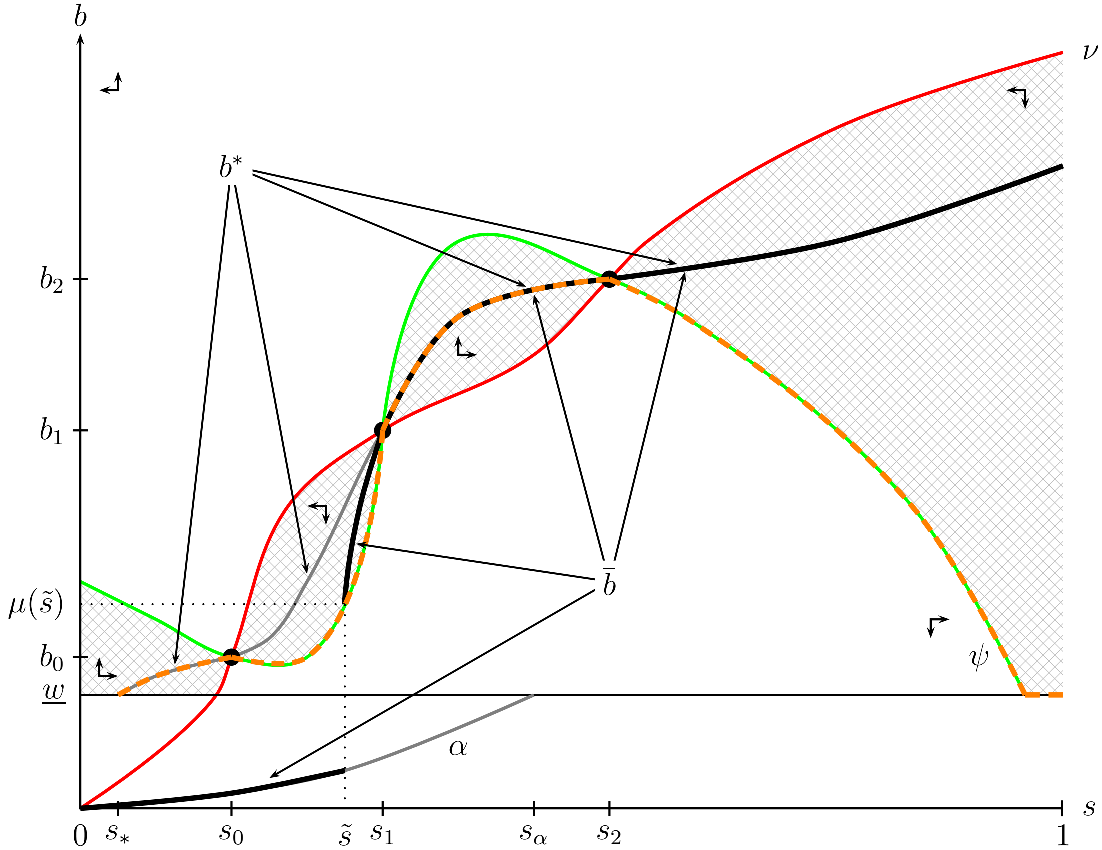


Figure B.2: Identification of $\bar{b}(s)$ when $\underline{w} > 0$. There are three critical points.

B.3 No Critical Points and Non-Canonical Equilibria

When $\underline{w} > 0$, system (B.1) may lack a critical point. In the absence of a critical point there is no "$b^*(\cdot)$ function" and Assumption 2 reduces to only its second part: If $\alpha(s_\alpha) = \underline{w}$ for some $s_\alpha \in (0, 1)$, then $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) > 0$. An amendment of the preceding analysis allows us to identify $\bar{b}(\cdot)$. First, let

$$\mu(s) := \begin{cases} \psi(s) & \text{if } \psi(s) \neq \varnothing \\ \underline{w} & \text{if } \psi(s) = \varnothing \end{cases}$$

on the domain $[s_\nu, 1]$ where $s_\nu := \nu^{-1}(\underline{w})$. This is the dashed curve in Figure B.3 running along the lower boundary of region R_1. At each point along this curve, there is an increasing solution of system (B.1) that can be extended to the boundary as an increasing function. The value $\tilde{s}$ solves the same indifference condition as in the baseline analysis given the redefined function $\mu(\cdot)$. As illustrated in Figure B.3, for $s < \tilde{s}$, $\bar{b}(s) = \alpha(s)$. For $s > \tilde{s}$, $\bar{b}(s)$ follows

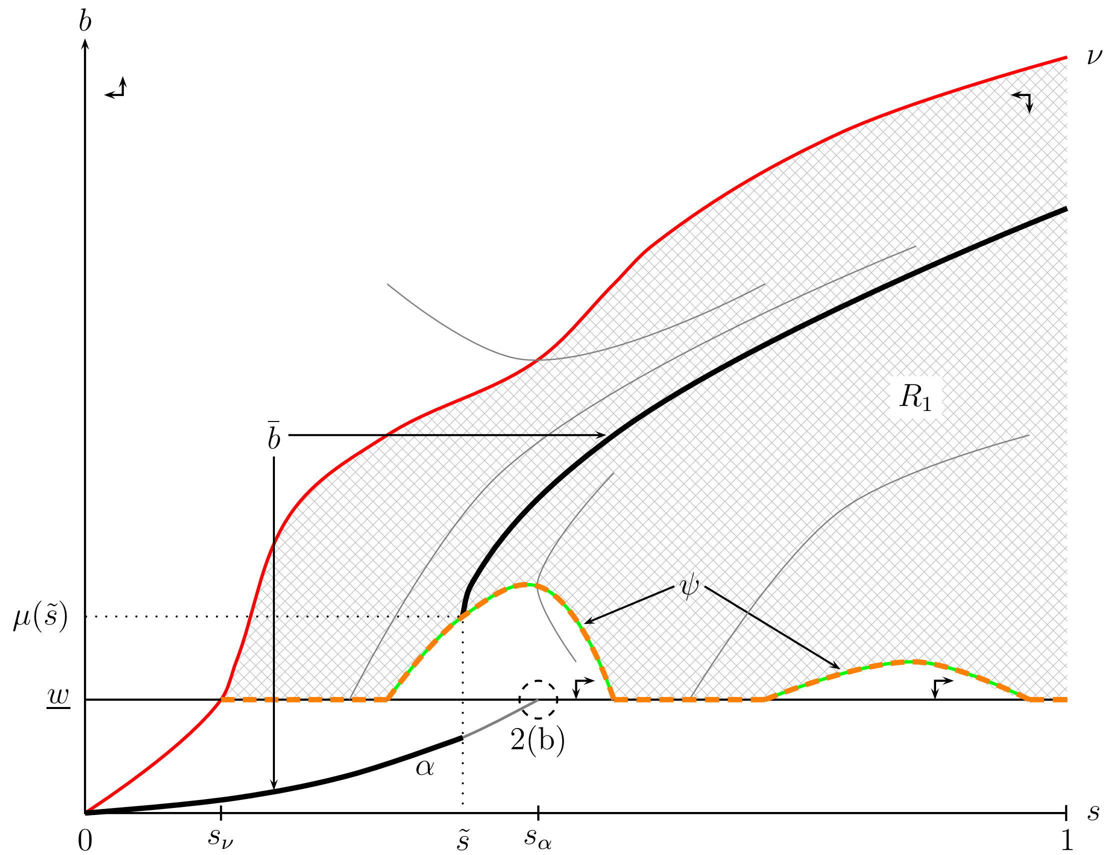


Figure B.3: Identification of $\bar{b}(s)$ when $\underline{w} > 0$. There are no critical points. Assumption 2(b) is satisfied.

the increasing solutions of system (B.1) passing through point $(\tilde{s}, \mu(\tilde{s}))$.

B.4 Type 2 Canonical Equilibria

When $\underline{w} > 0$, there may exist a canonical equilibrium $\beta(s, w) = \min\{\bar{b}(s), w\}$ where $\bar{b}(s)$ is a continuous function. Analysis of this case is presented in Figure B.4, which assumes the existence of one critical point. The same analysis applies if there are no critical points or multiple critical points. When $\bar{b}(s)$ is less than $\underline{w}$, $\bar{b}(s) = \alpha(s)$, the equilibrium strategy in the first-price auction in the absence of budget constraints. If there exists $s_\alpha \in (0, 1)$ such that $\alpha(s_\alpha) = \underline{w}$, then $\bar{b}(s)$ makes a continuous transition into the range of bids above $\underline{w}$ in a type 2 canonical equilibrium. The function $\bar{b}(s)$ follows the increasing solution of system (B.1) starting at point $(s_\alpha, \underline{w})$. For an increasing solution to exist at $(s_\alpha, \underline{w})$, a necessary condition is that $\eta(s_\alpha, s_\alpha) - \delta(\underline{w}, s_\alpha) \leq 0$. Note that $\tilde{s} = s_\alpha$ and $\bar{b}(\tilde{s}) = \underline{w} = \alpha(s_\alpha)$. Figure B.5 illustrates a type 2 canonical equilibrium.

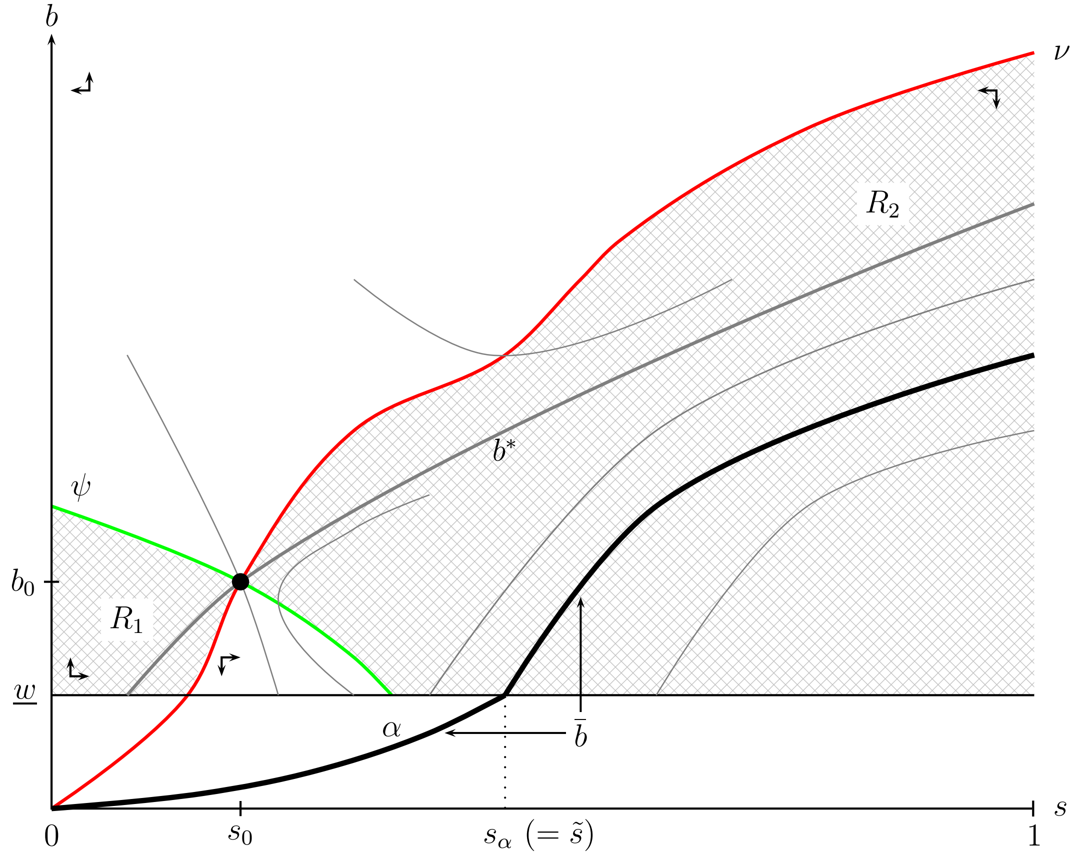
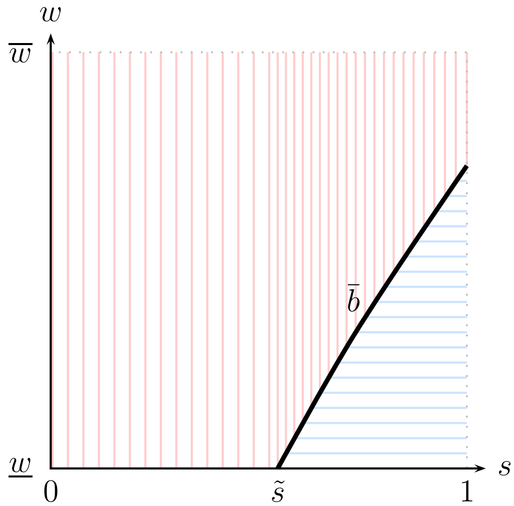


Figure B.4: Construction of a type 2 canonical equilibrium.

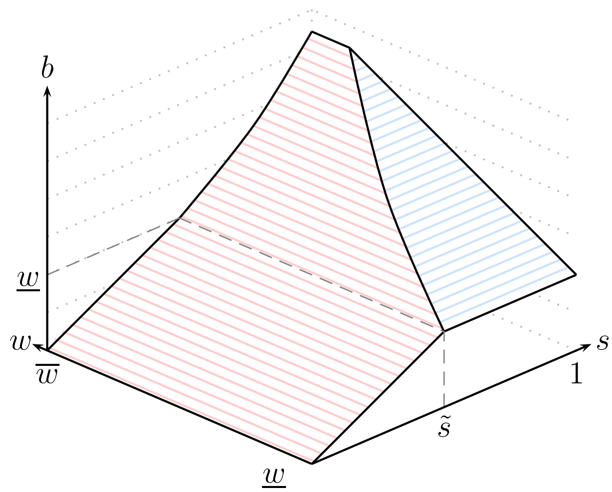
C Comparative Statics

The following example shows that skewing the budget distribution may induce higher or lower equilibrium bids by some unconstrained bidders.

Example C.1. Suppose there are two bidders with uniformly distributed value-signals on the unit interval. Suppose $v(s_i, s_j) = s_i + s_j$. Consider three different budget distributions on $[0, 2]$: $G_0(w) = w/2$, $G_1(w) = \sqrt{w/2}$, and $G_2(w) = w(4-w)/4$. Each satisfies Assumption 1′. The distribution G_0 likelihood ratio dominates G_1 and G_2. Given each budget distribution there is one critical point. Given the budget distribution G_0, the critical point occurs at $(s_0, b_0) = (0.1056, 0.2111)$. Given the budget distribution G_1, the critical point occurs at $(s_1, b_1) = (0.0864, 0.1728)$. Finally, given the budget distribution G_2, the critical point occurs at $(s_2, b_2) = (0.1264, 0.2528)$. By Theorem 1, the function defining the bid of an unconstrained bidder, $\bar{b}(s)$, passes through the critical point in each case. This implies that tightening the budget distribution from G_0 to G_1 induces a *lower* bid by an unconstrained bidder with a value-signal $s \approx 0.1$. Conversely, tightening the budget distribution from G_0



(a) Level sets of $\beta(s, w)$.



(b) Three dimensional sketch of $\beta(s, w)$.

Figure B.5: Illustration of a type 2 canonical equilibrium strategy $\beta(s, w) = \min\{\bar{b}(s), w\}$.

to G_2 induces a *higher* bid by an unconstrained bidder with a value-signal $s \approx 0.1$.

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