The Property Rights Theory of Production Networks

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Abstract

This paper investigates the formation of production and trading networks in an economy with general interdependencies and complex property rights. The right to exclude, a core tenet of property, grants asset owners a form of monopoly power that influences granular economic interactions. Equilibrium networks reflect the distribution of these ownership claims. Inefficient production networks may endure in equilibrium as firms multisource to mitigate hold-up risk. Short supply chains also reduce this risk, but may preclude the production of complex goods. A generalized Top Trading Cycles algorithm, applicable to a production economy, identifies equilibrium outcomes in the model. Such outcomes can be decentralized via a price system.

Keywords: Property Rights, Production Networks, Hold-up, Top Trading Cycles
JEL: C71, L14, D51

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1 Introduction

This paper proposes a property rights theory of production networks. We are motivated by the extensive interdependencies in the organization of economic activity. A firm's production process is a network threading together many inputs. Building an airplane, for example, combines engineering knowledge, parts from many suppliers, and plenty of human capital. Everything must fit for production, and the product, to get off the ground.

How does the distribution of property rights in a market affect the network structure of economic activity? The link, we argue, operates through a core tenet of property, *the right to exclude*. The United States Supreme Court has called the right to exclude "one of the most essential sticks in the bundle of rights that are commonly characterized as property"[1] and legal scholars have emphasized its significance since at least the eighteenth-century.[2] The right to exclude grants the owner(s) of an asset, including labor and human capital, a form of monopoly power that influences economic interactions at the granular level. By threatening to exclude others from desired goods or critical inputs—effectively a threat of hold-up—an agent can skew outcomes in his favor. Equilibrium production and trading networks balance the push and pull each agent exerts on others through his ownership claims to goods, machines, ideas, and human capital.

The push-and-pull we have in mind is subtle and often indirect. It exists because of the network of economic activity. If Alice can block Bob's access to apples and Bob can block Carol buying bananas, then Alice has indirect leverage when dealing with Carol through pressure she can exert on Bob. Accordingly, agents may structure their interactions to avoid others' exclusion power. This proclivity is prominent in production. Consider the airplane manufacturer from above. Many ideas are in the public domain and no one can be excluded from using them (e.g., Bernoulli's principle). But some are patented and wriggling out of the "patent thicket" is necessary.[3] Engines might come from multiple suppliers. This complicates the supply chain, but insures against being held up by critical source.

Do networks amplify or dampen the right to exclude's potency? What form do equilibrium production networks assume? And, what are the welfare implications? We investigate these questions theoretically in an economy with general interdependencies. A discrete set of goods lets us precisely map the economy's connections and transactions; otherwise, our setup is very

[1] Kaiser Aetna v. United States, 444 U.S. 164 (1979).

[2] William Blackstone described it in his *Commentaries on the Laws of England* (1765). Merrill (1998), Merrill and Smith (2001b), and Klick and Parchomovsky (2017) discuss the principle's significance in detail.

[3] Shapiro (2000) used the thicket metaphor to describe complex and overlapping patents.

general. Goods in our model can be rival or non-rival. They include primary goods, intermediate inputs, finished products, human-capital/labor inputs, laws or policies, and even ideas. Firms transform sets of inputs into sets of outputs. A firm's output can be an input for another firm. Agents own the economy's goods. Unlike traditional models of exchange or production, we allow for general ownership structures to emphasize the role of varied and complex property rights. A good may be owned privately, owned jointly by a coalition, or be part of the social endowment. An outcome in our model consists of a consumption allocation, which describes consumers' choices, and a production network, which defines firms' production decisions.

To analyze our economy, we examine its *exclusion core*. The exclusion core is a cooperative solution introduced by Balbuzanov and Kotowski (2019) to study discrete exchange economies with complex property rights, including cases with joint or contested ownership. Traditional solutions, such as the core, fail to fully capture the implications of agents exercising their exclusion rights.[4] Roughly, at an exclusion core allocation no coalition has the incentive and the ability to veto or block others' consumption by invoking its right to exclude. The latter is defined by the ownership of the economy's goods. We provide a self-contained introduction to the exclusion core solution in Section 3, though we stress here that it is uniquely suited for our analysis. First, the exclusion core's motivation emphasizes the exclusion aspect of property. It rests on a reinterpretation of agents' endowments in an economy as a distribution of exclusion rights, rather than as bundles of tradable things. And second, the concept is sensitive to the endogenous network of economic activity. Recalling the example above, it accounts for Alice's indirect leverage (via Bob) on Carol.

Our paper has three contributions. First, we generalize the exclusion core solution to a production economy.[5] Our generalization exploits inter-firm linkages—an output of firm f is an input for firm f'—to transmit threats of exclusion and hold-up. Thus, a production network extends a coalition's power beyond the goods that it owns directly, a fact that constrains equilibrium production networks. We investigate two versions of the exclusion core that differ in the ease with which coalitions can rewire production plans. The ex post exclusion core characterizes short-run outcomes where the underlying production network is rigid. The ex ante exclusion core characterizes long-run outcomes where production plans are fully flexible. Together, these benchmarks bound all plausible intermediate cases.

[4]Many solutions are also technically unappealing. For instance, the strong core is often empty, even in trivial instances of our model; the weak core fails to rule out implausible inefficiencies (Balbuzanov and Kotowski, 2019).

[5]In a working paper version of Balbuzanov and Kotowski (2019), we explored a prototype "production economy." That model of production differed significantly from the analysis proposed herein.

Second, we identify sufficient conditions for an economy's ex ante and ex post exclusion cores to be nonempty. Roughly, there must be "sufficient integration" in the ownership structure of each good's supply chain. This condition is much weaker than complete vertical integration and suggests how property rights can be arranged to limit hold-up risk. Ex ante exclusion core production networks may involve firms employing seemingly redundant inputs. By multisourcing, firms hedge against hold-up risk. Our model characterizes the operation of an economy reliant solely on property relations, as opposed to more elaborate contracts.

Our third contribution connects to a literature distinct from that of property rights or economic organization. Our model generalizes Shapley and Scarf's (1974) "house exchange" economy, famous for its introduction of the Top Trading Cycles (TTC) algorithm (attributed to David Gale). Our existence results rely on the first generalization of the TTC algorithm to a production economy. The novelties necessitated by production have intuitive economic interpretations in terms of simplifying supply chains and identifying firm boundaries. The monotonicity of the assignment process induced by the TTC algorithm stands behind our generalization and links the exclusion core with price equilibria in our model.

Our model emphasizes production in a networked economy. We argue that the resulting transaction network is closely tied to the distribution property rights, particularly the right to exclude. Importantly, the same underlying principles operate in many domains. Networks play a central role in international trade. These often bend to the exclusion power of governments (e.g., embargoes). In politics, legislators "produce" laws by bundling together policy proposals. The challenge is to neutralize the veto (i.e., exclusion) power of blocking coalitions. And, layers of "red tape," i.e., exclusion rights held by bureaucrats, allow for rent extraction. Adaptations of our model can tackle these applications, among many others.

Outline Section 2 introduces the model. Section 3 presents the exclusion core solution concept in the special case of an exchange economy. Section 4 generalizes the analysis to a production economy. Section 5 outlines the proof of our main theorem and introduces our generalization of the TTC algorithm. Section 6 discusses multisourcing and section 7 considers price equilibria. We conclude by relating our study to the literatures on property rights and production networks. For expositional ease, we simplify several features of our environment. Appendices A and B explain how our model embeds a larger class of production processes and preferences than emphasized in our main exposition. Appendix C provides an example of our algorithm's operation. Appendix D contains all proofs.

2 Model

An economy $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$ consists of agents, firms, goods, a preference profile, and an endowment system. $I := \{i_1, \ldots, i_n\}$ and $F := \{f_1, \ldots, f_m\}$ are finite sets of agents and firms, respectively. X is a finite set of goods, which can be very general. It may contain consumption goods (e.g., an apple) and production inputs, such as raw materials (e.g., a ton of iron ore), labor inputs (e.g., an hour of welding), and intangibles (e.g., a computer chip design). Each good x has a capacity $q_x \in \{1, \infty\}$ defining the number of agents and firms that can simultaneously consume it or use it as an input. A rival good, like the apple, has capacity one. A non-rival good, like the chip design, has infinite capacity.

Goods are partitioned into a set of primary goods and sets of goods produced by each firm, i.e., $X = X_0 \cup (\bigcup_{f \in F} X_f)$ where $X_f \cap X_{f'} = \varnothing$ for all $f \neq f'$. The set X_0 consists of primary goods that do not require production. The goods in set X_f are available if and only if they are produced by firm f using the eponymous production function $f : 2^X \to \{\varnothing, X_f\}$. Thus, a firm transforms sets of inputs into a set of (net) outputs.[6] For example, a computer manufacturer might transform labor and computer parts into finished computers and new technologies created by research and development. The former are consumer products; the latter are inputs for other firms. Each production function is monotone ($Z \subseteq Z' \implies f(Z) \subseteq f(Z')$) and satisfies the "no free lunch" property ($f(\varnothing) = \varnothing$).

For simplicity, we assume that production has a $0/1$ character. Either firm f produces nothing or all of X_f is created. To scale production, posit there are multiple copies of firm f. Each copy produces a version of X_f that differs only in an inconsequential way, such as the goods' serial numbers. This framework can embed production processes with decreasing and increasing returns to scale (see Appendix A).

Each agent i has a strict preference $\succ_i$ defined over X and an outside option $x_0 \notin X$ representing "no consumption." For simplicity, we assume each agent has unit demand. If $x \in X \cup \{x_0\}$ is preferred to $x' \in X \cup \{x_0\}$, then $x \succ_i x'$. We write $x \succeq_i x'$ if $x \succ_i x'$ or $x = x'$. In examples, we state an agent's preference by listing goods in his preferred order, i.e., $\succ_i : x, x', \ldots$. Unlisted items are inferior to the outside option. Appendix B outlines how to incorporate consumption of multiple goods into our model.

Property rights are a focus of our analysis. Accordingly, we posit a general framework subsuming private and public ownership as special cases. The economy's *endowment system* $\omega : 2^I \to 2^X$ identifies the goods owned by each coalition. It satisfies three basic properties.

[6] That is, firm f supplies X_f to the market. These goods are not used by f in production: $f(Z \cup X_f) = f(Z)$.

(A1) Agency: $\omega(\emptyset) = \emptyset$.

(A2) Monotonicity: $C' \subseteq C \implies \omega(C') \subseteq \omega(C)$.

(A3) Exhaustivity: $\omega(I) = X$.

Thus, ownership is restricted to agents or groups (A1), a coalition owns anything belonging to a sub-coalition (A2), and the grand coalition owns everything (A3). We further assume the endowment system satisfies

(A4) Weak non-contestability: For each $x \in X$, the set $C^x \coloneqq \bigcap_{C \in \{C' | x \in \omega(C')\}} C$ is not empty.

Assumption (A4) says that each good x has an essential set of principal owners C^x. Any group asserting ownership of x must count C^x among its members. The assumption relaxes Balbuzanov and Kotowski's (2019, p. 1667) non-contestability condition.

Many situations satisfy (A1)–(A4). If x is privately owned by i, then $x \in \omega(i)$ and $C^x = \{i\}$. If x is collectively owned by everyone, C^x is the grand coalition and $x \in \omega(C) \iff C = I$. If $x \notin \omega(C^x)$, then its principals require others' cooperation to exercise de facto control over x. Another interesting case arises when goods produced by the same firm have different principals (i.e., $x, y \in X_f$ but $C^x \neq C^y$). For example, seats at a concert might be controlled by the concert promoter but rights to the concert's recording might belong to a record label.

An *outcome* (μ, γ) consists of a consumption allocation and a production network. A *consumption allocation* $\mu \colon I \to X \cup \{x_0\}$ identifies the good consumed by each agent. Denote the goods consumed by coalition $C \subseteq I$ as $\mu(C) \coloneqq \bigcup_{i \in C} \mu(i)$.

A *production network* $\gamma \colon F \to 2^X$ identifies the inputs used ("consumed") by each firm. Denote the goods used by firms in set $G \subseteq F$ as $\gamma(G) \coloneqq \bigcup_{f \in G} \gamma(f)$ and, abusing notation, let $f_G(\gamma) \coloneqq \bigcup_{f \in G} f(\gamma(f))$ be these firms' aggregate (gross) output at γ.

An outcome (μ, γ) is *feasible* if (a) $\mu(I) \cup \gamma(F) \subseteq X_0 \cup f_F(\gamma) \cup \{x_0\}$, and (b) $|\{i \in I | x = \mu(i)\}| + |\{f \in F | x \in \gamma(f)\}| \leq q_x$ for all $x \in X$. Condition (a) says that every good that is consumed or used in production is a primary good, a good produced at γ, or the outside option. Condition (b) says that the number of agents and firms assigned an item cannot exceed its capacity. Analogously, a production network γ is feasible if (a) $\gamma(F) \subseteq X_0 \cup f_F(\gamma)$ and (b) $|\{f \in F | x \in \gamma(f)\}| \leq q_x$ for all $x \in X$. Let Γ be the set of feasible production networks.

To help fix ideas, Figure 1 illustrates two outcomes in an economy. There are two agents with endowments $\omega(i_1) = \{x_1, x_3, x_4\}$ and $\omega(i_2) = \{x_2, x_5\}$. In the figure, each good is "pointing" to its principal owner. Each good has unit capacity. Goods x_2 and x_4 are produced by firms f_1

(a) Outcome (μ, γ). All goods are available.

(b) Outcome (μ', γ'). Good x_2 is not available.

Figure 1: Example outcomes in a private-ownership economy.

and f_2, respectively. The other goods are primary goods. The firms' production functions are

$$f_1(Z) = \begin{cases} x_2 & \text{if } x_1 \in Z \text{ and } x_4 \in Z \\ \varnothing & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(Z) = \begin{cases} x_4 & \text{if } x_3 \in Z \text{ or } x_5 \in Z \\ \varnothing & \text{otherwise} \end{cases}.$$

Goods x_1 and x_4 are complements in the production of x_2. Goods x_3 and x_5 are substitutes in the production of x_4.

Figure 1(a) represents the outcome

$$\mu(i_1) = x_2 \qquad \mu(i_2) = x_3 \qquad \gamma(f_1) = \{x_1, x_4\} \qquad \gamma(f_2) = x_5.$$

In the figure, each agent is "pointing" to his assigned consumption good. Each produced good is "pointing" to the input(s) used in its production (dashed lines). In this case, x_4 (produced by f_2) is an input for x_2 (produced by f_1). Figure 1(b) represents the outcome

$$\mu'(i_1) = x_4 \qquad \mu'(i_2) = x_5 \qquad \gamma'(f_1) = \varnothing \qquad \gamma'(f_2) = x_3.$$

Now, x_4 is consumed by i_1 and cannot be used to make x_2. Thus, x_2 is unavailable.

Remark 1. Each outcome sketch in this paper follows Figure 1's conventions. A solid arrow links each agent with his assigned good, or each privately-owned good with its owner. A dashed arrow connects each produced good to the inputs used in its production.

3 Exchange Economies

We aim to characterize production networks and trading patterns in light of the economy's property rights distribution. Balbuzanov and Kotowski (2019) study a related question focusing only on an exchange economy. They argue that classic solution concepts, such as the core, fail to shed light on this problem in markets with complex property arrangements, as in our model. Accordingly, Balbuzanov and Kotowski (2019) introduce an alternative solution called the exclusion core. The exclusion core's foundation is a reinterpretation of endowments as a distribution of exclusion rights over the economy's goods. This section, based on Balbuzanov and Kotowski (2019), provides a self-contained exposition of this idea in an exchange economy. We pursue the production economy generalization in Section 4.

Consider the model introduced in Section 2, but without production. Only primary goods exist and an outcome reduces to a consumption allocation. A familiar intuition, dating to at least Edgeworth, is that an allocation that can be improved upon, or "blocked," by some coalition is unlikely to materialize or endure. Traditionally, a coalition can block an allocation in an exchange economy if it can reallocate the goods in its endowment in a way that its members prefer. However, for a reallocation to be possible, it is necessary to exclude non-coalition members from those goods, possibly harming them in the process. This more basic principle is at the heart of exclusion blocking.

Definition 1. Coalition $C \subseteq I$ can *directly exclusion block* the allocation μ if there exists an allocation σ such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \omega(C)$.

Two conditions must hold for a coalition to directly exclusion block an allocation μ. First, all coalition members must prefer an alternative allocation σ. And second, if an agent is harmed by the move from μ to σ, he must be excluded by this process from something in the coalition's endowment. Thus, a coalition has the interest and the ability, based on its exclusion rights, to veto the allocation μ. The *direct exclusion core* is the set of allocations that cannot be directly exclusion blocked by any nonempty coalition.

What indirect implications do exclusion rights entail? The case of Alice, Bob, and Carol from the introduction suggests that trade-related interdependencies can amplify an agent's de facto power. For example, consider a private-ownership economy with three agents and three goods. Their preferences are

$$\succ_{i_1} : x_2, x_1 \qquad \succ_{i_2} : x_1, x_3, x_2 \qquad \succ_{i_3} : x_1, x_2, x_3.$$

Figure 2(a) illustrates the allocation

$$\mu(i_1) = x_2 \qquad \mu(i_2) = x_3 \qquad \mu(i_3) = x_1.$$

Each agent is "pointing" to his assigned good and each good is "pointing" to its owner (i.e., $x_k \in \omega(i_k)$). At μ, i_1 and i_3 receive their favorite items; i_2 gets x_3, but prefers x_1. The allocation μ cannot be directly exclusion blocked, but can i_2 somehow shift the outcome in his favor?

(a) Allocation μ.

(b) Agent i_2 has indirect exclusion rights to x_1 given i_1's dependence on $x_2 \in \omega(i_2)$ at μ.

Figure 2: An allocation that can be destabilized through indirect exclusion.

Since $x_1 \notin \omega(i_2)$, i_2 lacks the direct exclusion rights to prevent i_3 from receiving x_1. Those rights belong to i_1. However, i_1 is assigned x_2, which belongs to i_2. Formally, $i_1 \in (\mu^{-1} \circ \omega)(i_2)$. Given this dependency, agent i_2 may press i_1 to prevent i_3's consumption of x_1 by threatening to exclude i_1 from x_2. Agent i_1 would reasonably accept this demand as x_2 is his favorite item. As illustrated in Figure 2(b), this fragility destabilizes μ—the matching μ is unlikely to endure.

Though Figure 2 examines a relatively simple case, its underlying logic generalizes. Everything proceeds by induction. Coalition C has direct exclusion rights to all things in its endowment, $\omega(C)$. At one step of influence, it has direct or indirect exclusion rights to all goods in $\omega(C_1)$, where $C_1 = C \cup (\mu^{-1} \circ \omega)(C)$, and at two steps, $\omega(C_2)$ where $C_2 = C_1 \cup (\mu^{-1} \circ \omega)(C_1)$. And so on. The limit defines the coalition's extended endowment, which describes its de facto exclusion power accounting for the network of interdependencies induced by exchange.

Definition 2. The *extended endowment of coalition C (in an exchange economy) at μ* is $\Omega(C|\omega,\mu) := \omega(\bigcup_{k=0}^{\infty} C_k)$ where $C_0 = C$ and $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for each $k \geq 1$.

The next definition repeats Definition 1, except $\Omega(\cdot|\omega,\mu)$ appears in part (b).

Definition 3. Coalition $C \subseteq I$ can *indirectly exclusion block* the allocation μ if there exists an allocation σ such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega,\mu)$.

The *exclusion core* is the set of allocations that cannot be indirectly exclusion blocked by any nonempty coalition. No coalition can improve upon an exclusion core allocation by invoking its direct or indirect exclusion rights.

Revisiting the example above, the allocation μ is not an exclusion core allocation. It can be indirectly exclusion blocked by i_2. The unique exclusion core allocation is

$$\sigma(i_1) = x_2 \qquad \sigma(i_2) = x_1 \qquad \sigma(i_3) = x_3,$$

which is illustrated in Figure 3(a). Whereas i_3 would like to claim x_1, the network of trades induced by σ protects i_1 and i_2 from i_3's (in)direct exclusion power (Figure 3(b)). Neither agent's allocation depends on i_3's endowment.

(a) Allocation σ.

(b) Agent i_3 lacks direct or indirect exclusion rights to x_1 at σ.

Figure 3: An allocation that cannot be destabilized through indirect exclusion.

The next theorem generalizes Theorem 1 of Balbuzanov and Kotowski (2019). It is implied by Theorem 3 (stated below), which concerns a more general model.

Theorem 1. *The exclusion core of an exchange economy where the endowment system satisfies (A1)–(A4) is not empty.*

Our interest in the exclusion core stems from its conceptual origins in property rights (the right to exclude in particular) and its sensitivity to the network of trades. However, it has many properties in an exchange economy that further elevate its appeal. For example, all exclusion core allocations are Pareto efficient.[7] In Shapley and Scarf's (1974) "house exchange" economy, a seminal special case, the exclusion core equals the strong core, a consensus selection.[8] Unlike the strong core, the exclusion core characterizes a generalization of the TTC algorithm in exchange economies with private and public ownership (Balbuzanov and Kotowski, 2019).

4 Production Economies

To generalize the exclusion core solution to a production economy, we rely on critical inputs to transmit exclusion and hold-up threats through a production network.

Definition 4. The set $Z \subseteq X$ is *critical* for the production of x given the production network γ if x can be produced by firm f with inputs $\gamma(f)$ and x cannot be produced by f with inputs $\gamma(f) \setminus Z$.

Any coalition controlling a critical set of inputs for x can exclude others from x by holding-up its production. This observation has significant consequences when coupled with the logic of Section 3. As motivation, consider the outcome illustrated in Figure 4. All goods are privately owned, except x_2, which is owned collectively. Consider i_1. He owns x_1, which is a critical input for x_2. Even though x_2 is collectively owned, i_1 can block its production through his

[7] Condition (b) in Definition 3 holds vacuously if everyone is (weakly) better off. Pareto efficiency follows.

[8] In Shapley and Scarf's model, there are n agents and n goods. Each agent i_k owns one good, $\omega(i_k) = \{x_k\}$, and $x \succ_{i_k} x_0$ for all $x \neq x_0$ and all i_k. See also our discussion in Section 5.1 concerning the TTC algorithm.

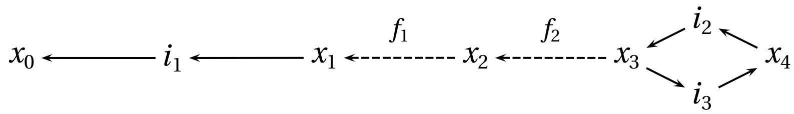


Figure 4: An outcome where i_1 has (in)direct exclusion rights to all four goods.

control of x_1. Going further, x_2 is a critical input for x_3. Thus, i_1 can block access to x_3 through his indirect control of a critical input. Since i_1 can block the production of x_3, by virtue of the reasoning from Section 3 he can press i_2 to exclude i_3 from x_4. In sum, we have argued that i_1 has indirect exclusion power over all four goods.

To formalize the preceding reasoning in the general case, we first derive a new characterization of a coalition's extended endowment in an exchange economy (Definition 2). The reformulation simplifies accounting as consumption (and, in the sequel, production) dependencies are traced through the economy.

Lemma 1. *If $\Omega(C|\omega,\mu)$ is the extended endowment of coalition C in an exchange economy at allocation μ, then*

$$\Omega(C|\omega,\mu) = \bigcup_{k=0}^{\infty} Z_k \tag{1a}$$

where $Z_0 = \omega(C)$ and

$$Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \tag{1b}$$

for each $k \geq 1$.

Next we generalize (1a) and (1b) accounting for critical production links. Let $\alpha_\gamma(Z)$ be the set of goods for which Z is a critical set of inputs at γ. If $x \in \alpha_\gamma(Z)$, then x is *directly reliant on Z at γ*. We append $\alpha_\gamma(\cdot)$ to each iteration of (1b). Thus, the *extended endowment of coalition C (in a production economy) at* (μ,γ) is

$$\Omega_\gamma(C|\omega,\mu) \coloneqq \bigcup_{k=0}^{\infty} Z_k \tag{2a}$$

where $Z_0 \coloneqq \omega(C)$ and

$$Z_k \coloneqq Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1}) \tag{2b}$$

for each $k \geq 1$. In a production economy, a coalition acquires (indirect) exclusion rights to a good through both consumption and production dependencies.

We can now define exclusion blocking in a production economy by plugging $\Omega_\gamma(\cdot|\omega,\mu)$ into part (b) of Definition 3. However, one subtlety remains. It is not obvious when or if a blocking coalition can change a firm's operation. For example, suppose firm f produces $\{x, y\}$

and coalition C includes all members of C^x but not C^y. Can the coalition tell the firm to shut down? When can a coalition demand that a shuttered firm start production? Similar questions arise when defining the (regular) core in a production economy and there are no unequivocal answers.[9] Accordingly, we propose two definitions that bracket most, if not all, plausible proposals. The "ex post" exclusion core posits agents take firms' production plans as given. The "ex ante" exclusion core allows blocking coalitions to change all production plans freely. We investigate each variant in turn.

4.1 The Ex Post Exclusion Core

Definition 5. Coalition $C \subseteq I$ can *ex post exclusion block* the outcome (μ, γ) if there exists a feasible outcome (σ, γ) such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega, \mu)$.

The *ex post exclusion core* is the set of feasible outcomes that cannot be ex post exclusion blocked by any nonempty coalition. In Definition 5, a blocking coalition takes the production network as given. The notion captures a short-run disruption wherein firms cannot easily retool or establish new production relationships. For intuition, imagine firms initially cement inter-firm links by committing to a production network. Subsequently, agents coordinate on a consumption allocation. If all feasible consumption allocations can be ex post exclusion blocked, the production network is unambiguously incredible. Anticipation of ex post opportunism may even curse all parties to settle for a "no production" outcome. In fact, if the economy's endowment system satisfies (A1)–(A4), there always exists an ex post exclusion core outcome with an empty production network, $\gamma(f) = \varnothing$ for all $f \in F$. Without production, only primary goods exist and Theorem 1 applies.

When does a nontrivial production network γ support an ex post exclusion core outcome? To answer this question, we first examine the production network's critical connections. Recall that $\alpha_\gamma(\cdot)$ identifies the goods directly reliant on any set of inputs. Iterating this mapping gives the indirectly reliant goods. For any $Z \subseteq X$, the set of goods *indirectly reliant on Z at γ* is

$$\lambda_\gamma(Z) := \bigcup_{k=0}^{\infty} A_k$$

[9]Kreps (2013, pp. 369–370) discusses this problem in the case of the Arrow-Debreu model. He provides five definitions of the core in a production economy that differ in the degree to which blocking coalitions are able to draw upon or change firms' production plans.

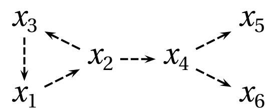


Figure 5: The production network γ. If $x_k \dashrightarrow x_\ell$, then x_k is produced using x_ℓ.

where $A_0 \coloneqq Z$ and $A_k \coloneqq A_{k-1} \cup \alpha_\gamma(A_{k-1})$ for each $k \geq 1$. Let

$$\Lambda_\gamma(x) \coloneqq \left\{ Z \in 2^X \,\middle|\, x \in \lambda_\gamma(Z) \;\&\; x \notin \lambda_\gamma(Z') \;\forall Z' \subsetneq Z \right\}.$$

If $Z \in \Lambda_\gamma(x)$, then x is directly or indirectly reliant on Z and no proper subset of Z has this property. When $Z = \{z\}$, we suppress the braces by writing $z \in \Lambda_\gamma(x)$.

To better understand $\lambda_\gamma(\cdot)$ and $\Lambda_\gamma(\cdot)$, first note that (by definition) $x \in \lambda_\gamma(x)$ and $x \in \Lambda_\gamma(x)$ for all $x \in X$, including primary goods and goods that are not produced at γ. Second, $\lambda_\gamma(\cdot)$ and $\Lambda_\gamma(\cdot)$ have simple graph-theoretic characterizations when γ is efficient. The input plan $\gamma(f)$ is *efficient for firm* f if there is no $Z \subsetneq \gamma(f)$ that assures f the same output as $\gamma(f)$. The production network γ is *efficient* if $\gamma(f)$ is efficient for every firm. When γ is efficient, every element of $\Lambda_\gamma(x)$ is a singleton (the proof is analogous to that of Lemma 3(a)-(b) in Appendix D). Now consider the production network γ illustrated in Figure 5. Each good is "pointing" to the inputs used in its production, if any. If γ is efficient, then $y \in \lambda_\gamma(x)$ if and only if there is a path in the graph from y to x. For example, $\lambda_\gamma(x_4) = \{x_1, x_2, x_3, x_4\}$.[10] Conversely, $y \in \Lambda_\gamma(x)$ if and only if there is a path in the graph from x to y. Thus, $\Lambda_\gamma(x_4) = \{x_4, x_5, x_6\}$.[11]

The interaction between critical inputs and the economy's endowment system is at the heart of the following theorem. We discuss its interpretation below.

Theorem 2. *Let $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$ be an economy satisfying (A1)–(A4). Let $\gamma \in \Gamma$ and suppose*

$$C(x|\gamma) \coloneqq \bigcap_{Z \in \Lambda_\gamma(x)} \left(\bigcup_{z \in Z} C^z \right) \neq \varnothing \tag{3}$$

for all $x \in X$. There exists an allocation μ such that (μ, γ) is an ex post exclusion core outcome.

Condition (3) holds for all primary goods and for all goods that are not produced at γ.[12] Thus, (3) only has bite if x is produced at γ. Since $x \in \Lambda_\gamma(x)$, $C(x|\gamma) = C^x \cap (\bigcap_{Z \in \Lambda_\gamma(x) \setminus \{x\}} (\bigcup_{z \in Z} C^z))$.

[10] Per convention, a single vertex is a length zero path that begins and ends at itself. Thus, $x_4 \in \lambda_\gamma(x_4)$.

[11] To simplify, we omit the braces for singleton elements. A complementary representation of $\lambda_\gamma(\cdot)$ and $\Lambda_\gamma(\cdot)$ is available in terms of the economy's "Leontief inverse" matrix, which can be expressed as an infinite sum of powers of the production network's adjacency matrix. Details of this interpretation are available from the authors upon request.

[12] If x is a primary good or a good that is not produced at γ, then no set of inputs is critical for x. Therefore, $x \in \lambda_\gamma(Z) \iff x \in Z$, which implies $\Lambda_\gamma(x) = \{x\}$. Thus, (3) reduces to $C^x \neq \varnothing$, which holds by (A4).

The first term, C^x, is good x's principals. The second term concerns the intermediate goods in x's supply chain. Thus, (3) holds at x when the principals of each critical set of (indirect) inputs overlap with the principals of x. Intuitively, there is "sufficient integration" in the supply chain's ownership structure to neuter any hold-up risk due to misaligned interests.

Three observations are pertinent. First, condition (3) *does not* imply full integration of the supply chain's ownership. It is a much weaker postulate. If z is the only critical input for x and $C^x = \{i_1, i_2\}$ and $C^z = \{i_2, i_3\}$, then the supply chain is sufficiently integrated. Even though i_2 does not own x or z outright, he has sufficient power to neutralize hold-up risk. Second, condition (3) is independent of agents' preferences. Therefore, it is predicated on a "worst-case" scenario with misaligned interests. If agents' preferences do not conflict, an ex post exclusion core outcome obtains under weaker institutional conditions. Third, the ex post exclusion core becomes larger as the economy shifts away from private property or toward a less efficient production network. Proofs of the following propositions are omitted. They follow from the monotonicity of $\Omega_\gamma(\cdot|\omega, \mu)$ in ω and γ.

Proposition 1. *Let (μ, γ) be an ex post exclusion core outcome in $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$. Consider $\mathscr{E}' = \langle I, F, X, \succ, \omega'\rangle$ where $\omega'(C) \subseteq \omega(C)$ for all $C \subseteq I$. The outcome (μ, γ) belongs to the ex post exclusion core of $\mathscr{E}'$.*

Proposition 2. *Let (μ, γ) be an ex post exclusion core outcome in $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$. Consider the feasible outcome (μ, γ') where $f_F(\gamma) = f_F(\gamma')$ and $\gamma(f) \subseteq \gamma'(f)$ for all $f \in F$. The outcome (μ, γ') belongs to the ex post exclusion core of $\mathscr{E}$.*

In Proposition 2, the production network γ is *more efficient than* γ' since it assures the same output with fewer inputs. Intuitively, each input's importance at γ is amplified and its owners have greater (indirect) exclusion power, thus narrowing the ex post exclusion core's size.

4.2 The Ex Ante Exclusion Core

Definition 6. Coalition $C \subseteq I$ can *ex ante (indirectly) exclusion block* the outcome (μ, γ) if there exists a feasible outcome (σ, ψ) such that (a) $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and (b) $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega, \mu)$.

Definition 6 differs from Definition 5 in one way—the production network can change freely. The *ex ante exclusion core* is the set of feasible outcomes that cannot be ex ante exclusion blocked by any nonempty coalition. Every outcome in the ex ante exclusion core is

Pareto optimal[13] and belongs to the ex post exclusion core.

The ex ante exclusion core can be empty without further restrictions on the environment. We focus on a class of economies where each firm has a unique efficient production plan. In our model, this is equivalent to the case of input complementarities. Firm f has a *Leontief production function* if there exists a set of inputs W_f, $W_f \cap X_f = \varnothing$, such that

$$f(Z) = \begin{cases} X_f & \text{if } Z \supseteq W_f \\ \varnothing & \text{otherwise} \end{cases}. \tag{4}$$

Production processes with complementarities are salient at the granularity and disaggregation implicit in our model. Even minor input substitutions can result in a distinct final product if goods are defined sufficiently narrowly. Production complementarities are important for trade patterns (Grossman et al., 2005) and economic growth (Kremer, 1993; Jones, 2011). Both topics are natural applications of our theory emphasizing exclusion rights and hold up.

Theorem 3. *Let $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$ be an economy where ω satisfies (A1)–(A4),*

(B1) each firm has a Leontief production function,

(B2) there exists a $\overline{\gamma} \in \Gamma$ such that $X = X_0 \cup f_F(\overline{\gamma})$, and

(B3) $C(x|\overline{\gamma}) \neq \varnothing$ for all $x \in X$.

There exists an ex ante exclusion core outcome in $\mathscr{E}$.

Assumption (B1) was discussed above. Assumption (B2) ensures that it is feasible to produce all goods. This assumption is analogous to Assumption V in Arrow and Debreu (1954, p. 280) that assures an excess supply of all goods can be achieved by some production plan. It *does not* imply that all goods are available for consumption or that $\overline{\gamma}$ is the production network at an exclusion core outcome. Assumption (B3) is the same as (3) in Theorem 2, but evaluated only at $\overline{\gamma}$. Theorem 1 is a corollary of Theorem 3; (B1)–(B3) are moot or trivially satisfied if all goods are primary.

[13] If (μ, σ) is not Pareto optimal, then there exists a feasible outcome (σ, ψ) such that $\sigma(i) \succeq_i \mu(i)$ for all $i \in I$ and $\sigma(i') \succ_{i'} \mu(i')$ for some $i' \in I$. Thus, (μ, γ) can be ex ante exclusion blocked by the coalition $C = \{i'\}$ with (σ, ψ). Condition (b) in Definition 6 holds vacuously—there is no agent j for whom $\mu(j) \succ_j \sigma(j)$.

5 Proof of Theorem 3

This section explains the key economic insights from the proof of Theorem 3 (see Appendix D). The proof has two parts. Part I constructs an ex ante exclusion core outcome in an acyclic economy (defined below). Part II extends this result to any economy satisfying the hypotheses of Theorem 3 by identifying a mapping between it and a corresponding acyclic economy.

5.1 Part I — Top Trading Cycles and Acyclic Economies

To prove Theorem 3, we first construct an outcome in an economy satisfying the theorem's hypotheses using a generalized TTC algorithm. Before tackling the details, we highlight the guiding principle behind our generalization. In Shapley and Scarf's (1974) model, the classic TTC algorithm identifies the unique exclusion core outcome (Balbuzanov and Kotowski, 2019). In this special case, there is no production and each agent owns exactly one item. The TTC algorithm proceeds as follows. Each good "points" to its owner and each agent "points" to his favorite item. Necessarily, there is a cycle of alternating goods and agents. Each agent in the cycle leaves the market with the good to which he is pointing. Then, the process iterates with each remaining agent pointing to his favorite still-available item. Notice that with each iteration the set of available goods shrinks and goods never return to the market. In step 1, every good is available. At step 2, all goods except those cleared in step 1 are available. And so on. Theorem 3's assumptions let us define a similarly-monotone assignment process. Successive cycles define agents' consumption allocations and firms' input use. A novel "cycle trimming" procedure arbitrates agent-firm conflicts to ensure compatibility of consumption assignments and the supply chains necessary for production. We explain this feature's economic intuition below.

The algorithm requires two preliminary definitions. The production network $\gamma \in \Gamma' \subseteq \Gamma$ is *maximal in* Γ' if there does not exist a $\gamma' \in \Gamma'$ such that $\gamma' \neq \gamma$ and $\gamma'(f) \supseteq \gamma(f)$ for every f. For any $X' \subseteq X$ and $F' \subseteq F$, the production network $\gamma\colon F' \to 2^X$ is (X', F')-*feasible* if (a) $\gamma(F') \subseteq X' \cup f_{F'}(\gamma)$ and (b) $|\{f \in F' | x \in \gamma(f)\}| \leq q_x$ for all $x \in X' \cup f_{F'}(\gamma)$.

Algorithm 1. Given $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$, construct the outcome (μ, γ) in a series of steps. In step $t \geq 1$, the algorithm proceeds as follows with inputs (I^t, F^t, X_0^t). I^t is the set of unassigned agents, F^t is the set of unassigned firms, and X_0^t is the set of primary and previously-produced goods with remaining capacity. Let $I^1 := I$, $F^1 := F$, and $X_0^1 := X_0$.

Step t Let $X^t \coloneqq X_0^t \cup f_{F^t}(\gamma^t)$ where $\gamma^t \colon F^t \to 2^X$ is the maximal, efficient (X_0^t, F^t)-feasible production network.[14] Construct a directed graph as follows. Let $I^t \cup X^t \cup \{x_0\}$ be the set of nodes. Draw an arc from each $i \in I^t$ to the $\succ_i$-maximal element in $X^t \cup \{x_0\}$. If $x \in X^t$ and $C(x) \coloneqq C(x|\gamma^t) \cap I^t \neq \varnothing$, draw an arc from x to the lowest-index agent in $C(x)$.[15] Else if $C(x) = \varnothing$, draw an arc from x to the lowest-index agent in I^t.

If there is a link from $i \in I^t$ to x_0, then define $\mu(i) = x_0$, $\tilde{I}^t = \{i\}$, and $\tilde{X}^t = \varnothing$. Update the algorithm's inputs—$I^{t+1} \coloneqq I^t \setminus \tilde{I}^t$, $F^{t+1} \coloneqq F^t$, $X_0^{t+1} \coloneqq X_0^t$—and proceed to step $t+1$.

Otherwise, from each $i \in I^t$ there is a link to some $x \in X^t$. Since each agent is "pointing" to a good and each good is "pointing" to an agent, there exists a cycle of alternating agents and goods. (A cycle may be formed by one agent and one good.) If there are multiple cycles, they are disjoint and we may focus on any of them. Let K be that cycle.

Given K, pick any two distinct goods $x, y \in K \cap X^t$ such that $x \in \Lambda_{\gamma^t}(y)$. If there are no such goods, continue to ($\star$) below. Otherwise, iterate the following operation to define a new cycle until it contains no distinct goods x and y such that $x \in \Lambda_{\gamma^t}(y)$.

> *Cycle Trimming.* Since x and y belong to the same cycle, these goods are pointing to different agents. Say, $x \to i$ and $y \to j$. Delete the arc from x to i and draw a new arc from x to j, thus defining the new cycle $K' \subseteq K$. The new cycle K' does not contain good y or agent i.

($\star$) Given the identified cycle K, perform the following assignments.

(a) If $i \to x$ in the cycle, set $\mu(i) = x$. Let $\tilde{I}^t$ be the set of agents whose assignment has just been defined. Let $I^{t+1} \coloneqq I^t \setminus \tilde{I}^t$ be the set of agents for whom $\mu(\cdot)$ is yet undefined.

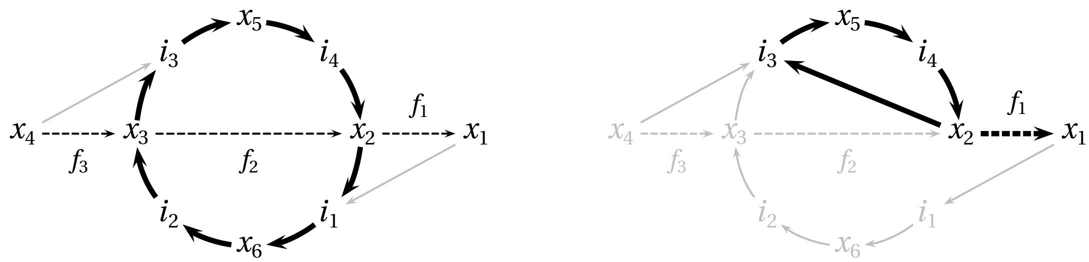
(b) Each $x \in (\bigcup_{z \in K \cap X^t} \Lambda_{\gamma^t}(z)) \setminus X_0^t$ is either a produced good that is assigned to an agent in (a) or a produced good that is an (indirect) input for a good that is assigned to an agent in (a). This good's producer, say f, belongs to F^t. Accordingly, for each such firm define $\gamma(f) = \gamma^t(f)$. Let $\tilde{F}^t$ be the set of firms whose input assignment has just been defined.

(c) Let $\tilde{X}^t \subseteq \mu(\tilde{I}^t) \cup \gamma(\tilde{F}^t)$ be the set of goods assigned to agents or firms in parts (a) and (b) whose capacity has been depleted. Let $X_0^{t+1} \coloneqq (X_0^t \cup f_{\tilde{F}^t}(\gamma^t)) \setminus \tilde{X}^t$ be the set of primary goods or goods produced up to step t with remaining capacity.

(d) If $F^t \setminus \tilde{F}^t = \varnothing$, set $F^{t+1} = \varnothing$. Otherwise, define $\hat{\gamma}^t$ as the maximal, efficient $(X_0^{t+1}, F^t \setminus \tilde{F}^t)$-feasible production network. Let $\hat{F}^t \coloneqq \{f \in F^t \setminus \tilde{F}^t | \hat{\gamma}^t(f) = \varnothing\}$ be the set of remaining

[14] Lemma 5 in Appendix D shows that γ^t exists and is uniquely defined.

[15] Recall that $I = \{i_1, i_2, \dots, i_n\}$. We select the agent with the lowest index number for simplicity of exposition.



(a) The cycle before trimming. (b) The cycle after trimming.

Figure 6: The cycle trimming procedure.

firms that are assigned no inputs at $\hat{\gamma}^t$. For each $f \in \hat{F}^t$, assign $\gamma(f) = \emptyset$ and denote the set of these firms' (henceforth, not produced) outputs by $\hat{X}^t$. Let $F^{t+1} := F^t \setminus (\tilde{F}^t \cup \hat{F}^t)$ be the set of firms for which $\gamma(\cdot)$ is still undefined.

Given the newly defined parameters, $(I^{t+1}, F^{t+1}, X_0^{t+1})$, proceed to step $t+1$.

The above procedure continues until $I^t = \emptyset$. At this point, $\mu(\cdot)$ has been defined for all $i \in I$ and $\gamma(\cdot)$ has been defined for all $f \in F \setminus F^t$. For any remaining $f \in F^t$, set $\gamma(f) = \emptyset$.

Since there is a finite number of agents and at least one agent is assigned in each step, Algorithm 1 terminates in a finite number of steps. An example in Appendix C illustrates the algorithm's operation. Next, we address two topics related to the feasibility of the outcome constructed by Algorithm 1.

Cycle Trimming A distinctive feature of Algorithm 1 is the cycle trimming procedure. To understand this step's importance consider Figure 6(a), which illustrates a cycle arising at some step t. Each good is pointing to an agent and each agent is pointing to a good. Suppose, as illustrated by the dashed arrows, there is a supply chain at γ^t—x_4 is reliant on x_3, x_3 is reliant on x_2, etc. If x_2 has capacity one, the cyclic assignment is infeasible. If x_3 is produced for i_2's consumption, it needs x_2 as an input. Thus, x_2 cannot be consumed by i_4, as required by the cycle. The cycle trimming procedure resolves this conflict. Since $x_2 \in \Lambda_{\gamma^t}(x_3)$, the procedure has x_2 point to i_3 instead of i_1. A new cycle without x_3 is formed (Figure 6(b)). The new cycle's implied allocation is feasible.

The economics of the cycle trimming procedure are interesting. Intuitively, the operation cuts out goods with "long" supply chains. A preference for "short" supply chains is warranted given the underlying premise of exclusion blocking. If good x is a critical (indirect) input for y, then the owners of x have leverage over good y's assignment. Expansive supply chains introduce many input dependencies, thereby amplifying hold-up risk. Thus, the trading protocol steers the market toward less-complex and more robust production arrangements.

Production Cycles Cycle trimming contributes to the feasibility of Algorithm 1's output. However, it is not sufficient. Cyclic input-output relationships among firms remain a concern. For example, a coal mine supplies a power plant and simultaneously uses the generated electricity. Algorithm 1's output may be infeasible in such cases.

A simple solution to the production-cycles problem is to (temporarily) assume it away. Consider an economy $\mathscr{E}$ where every firm's production function satisfies (4). The economy's *input network* is a directed graph Φ where the set of nodes is F and there is a directed edge from $f \in F$ to $f' \in F$ if and only if there exists $x \in W_f \cap X_{f'}$. Thus, an output of firm f' is an input for firm f. The input network Φ is *acyclic* if for all $f, f' \in F$ such that there is a path in Φ from f to f', there is no path from f' back to f. The economy $\mathscr{E}$ is acyclic if its input network is acyclic. Acyclic input networks describe supply chains with well-defined "upstream" and "downstream" firms. Acyclicity is commonly assumed in studies of production and trading networks (Ostrovsky, 2008; Manea, 2018; Kotowski and Leister, 2019). If $\mathscr{E}$ is acyclic and satisfies the hypotheses of Theorem 3, then Algorithm 1 identifies a feasible, ex ante exclusion core outcome (Lemmas 9 and 10 in Appendix D).

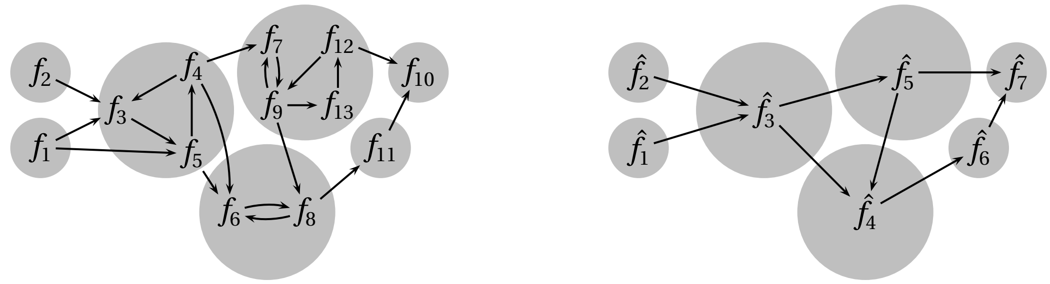
5.2 Part II — Condensation and Firm Boundaries

To extend our analysis beyond the acyclic case, we rely on the graph-theoretic concept of a *condensation* (Bondy and Murty, 2008, pp. 91–92). Consider an input network Φ. The firms f and f' are *strongly connected* if there is a path in Φ from f to f' and from f' back to f. A *strongly connected component* of Φ is a set of firms F_k such that each pair $f, f' \in F_k$ are strongly connected and F_k is not a proper subset of any other set of firms that are strongly connected.[16] Figure 7(a) illustrates an input network with thirteen firms and seven strongly connected components. Within each component, there is either a single firm or an input-output cycle among the constituent firms. Figure 7(b) presents the network's condensation, which is formed by contracting the nodes in each strongly connected component into a single node while preserving any external links. A directed graph's condensation is a directed acyclic graph. An acyclic graph's condensation is the graph itself.

We adapt the idea of a condensation of a directed graph to define the *condensation of an economy* $\mathscr{E}$, denoted as $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega} \rangle$ (Definition 8 in Appendix D). Intuitively, the concept involves "merging" the firms in each strongly connected component of the economy's input network.[17] For example, in Figure 7 $\{f_3, f_4, f_5\}$ becomes $\hat{f}_3$. The merged firm's "internal"

[16] A firm that is not strongly connected to any other firm forms a strongly connected component by itself.

[17] There are some additional subtleties. Endowments and primary goods need to be defined in $\hat{\mathscr{E}}$ as well.



(a) An input network with thirteen firms. (b) The input network's condensation.

Figure 7: The condensation of an input network.

input-output dependencies cancel out leaving only its "external" factor demand/supply relationships intact. $\hat{\mathscr{E}}$ is an acyclic economy that satisfies the hypotheses of Theorem 3. It has an ex ante exclusion core outcome that implies existence of an ex ante exclusion core outcome in the original economy $\mathscr{E}$ (see the proof of Theorem 3 in Appendix D).

The economic interpretation of a condensation depends on the granularity captured by the original economy $\mathscr{E}$. At the (extreme) micro-level, strongly connected components define firm boundaries. Posit that each "firm" in $\mathscr{E}$ is a discrete production task. Closely-related tasks tend to be integrated within a single organization, if only to save on coordination and transaction costs (Coase, 1937). A strongly connected component of $\mathscr{E}$ identifies a set of technologically co-dependent tasks, say $F_k = \{f, f', \ldots\}$, and the condensed firm $\hat{f}_k$ coordinates the tasks' execution. The interactions within firm $\hat{f}_k$ are a black box. Similar logic applies at a higher level of aggregation with each $f \in F_k$ representing a production plant of some larger entity $\hat{f}_k$. Functional entities among co-reliant organizations may also be interpreted as "condensed firms" in our model. Examples include patent pools and joint ventures.

6 Multisourcing

What kinds of production networks are consistent with ex ante exclusion core outcomes? Under the hypotheses of Theorem 3, each ex ante exclusion core outcome is consumption-equivalent to one with an efficient production network.

Proposition 3. *Let $\mathscr{E}$ be an economy satisfying (A1)–(A4) and (B1) with ex ante exclusion core outcome (μ, γ). There exists an efficient production network $\hat{\gamma}$ such that $(\mu, \hat{\gamma})$ is an ex ante exclusion core outcome in $\mathscr{E}$.*

In general, however, an inefficient production network may be necessary to support a particular consumption allocation as an ex ante exclusion core outcome. Engaging redundant inputs can insulate a firm from hold-up threats. The supply-chain practice of multisourcing demonstrates this idea in practice.

Example 1. Let $I = \{i_1, i_2, i_3\}$ and $X = \{x_1, \ldots, x_5\}$. Good x_1 is produced according to the production function

$$f(Z) = \begin{cases} x_1 & \text{if } x_2 \in Z \text{ or } x_3 \in Z \\ \varnothing & \text{otherwise} \end{cases}.$$

The remaining goods are primary goods. The agents' preferences are

$$\succ_{i_1}: x_1, x_4 \qquad \succ_{i_2}: x_1, x_4 \qquad \succ_{i_3}: x_1, x_5.$$

The endowment system is $\omega(i_1) = \{x_4, x_5\}$, $\omega(i_2) = \{x_2\}$, and $\omega(i_3) = \{x_3\}$. Good x_1 is owned collectively: $x_1 \in \omega(C) \iff C = I$.

This economy has several ex ante exclusion core outcomes. Of particular interest is

$$\mu(i_1) = x_1 \qquad \mu(i_2) = x_4 \qquad \mu(i_3) = x_5 \qquad \gamma(f) = \{x_2, x_3\}.$$

where x_1 is produced using *both* inputs. Such multisourcing insulates i_1 from challenges by i_2 and i_3. Both want to claim x_1, but neither has the ability to do so.

7 Price Equilibrium

A connection exists between trading cycle algorithms and price equilibria (Shapley and Scarf, 1974; Richter and Rubinstein, 2015). In our model, feasible outcomes identified by Algorithm 1 can be decentralized via the price system. We let $p \in \mathbb{R}_+^{|X \cup \{x_0\}|}$ denote a price vector; p_x is the price of good x. The next definition adapts a definition of Debreu (1959, p. 93).

Definition 7. The outcome (μ^*, γ^*) is an *equilibrium relative to the price system* p^* if

(a) for all $i \in I$, $x \succ_i \mu^*(i) \implies p^*_x > p^*_{\mu^*(i)}$;

(b) for all $f \in F$, $\gamma^*(f) \in \arg\max_{Y \subseteq X} \sum_{x \in f(Y)} p^*_x - \sum_{z \in Y} p^*_z$; and,

(c) the outcome (μ^*, γ^*) is feasible.

Conditions (b) and (c) are standard requirements concerning profit-maximization by firms and feasibility, respectively. Condition (a) describes consumer behavior and requires elaboration. Recall that an endowment system $\omega(\cdot)$ defines a distribution of exclusion rights, which can be very general. When agents lack a private endowment or goods are controlled collectively, it is unclear how to define personal budget sets without further ad hoc assumptions.[18] Definition 7(a) sidesteps this difficulty. If agent i prefers x to his consumption choice $\mu^*(i)$, then x must be more expensive than $\mu^*(i)$. This is an immediate implication of utility maximization subject to a budget constraint.[19]

Proposition 4. *Let $\mathscr{E}$ be an economy satisfying (A1)–(A4) and (B1)–(B3) where each good has capacity one. There exists an ex ante exclusion core outcome (μ^*, γ^*) and a price vector p^* such that (μ^*, γ^*) is an equilibrium relative to p^*.*

Corollary 1. *Let $\mathscr{E}$ be an acyclic economy satisfying (A1)–(A4) and (B1)–(B3) where each good has capacity one. If (μ^*, γ^*) is an outcome identified by Algorithm 1, then there exists a price vector p^* such that (μ^*, γ^*) is an equilibrium relative to p^*.*

If there are goods with infinite capacity (i.e., public goods), then it is straightforward to adapt the proof of Proposition 4 to construct an equilibrium with personalized prices (i.e., the Lindahl equilibrium analogue of Definition 7).

8 Related Literature and Concluding Remarks

Our study contributes to the literatures on property rights and economic organization, and trading and production networks. We conclude by discussing each contribution.

Property Rights and Economic Organization The interplay between property rights and economic organization has received considerable attention (Coase, 1960; Demsetz, 1967). Most recent treatments of this question build upon the incomplete contracts framework of Grossman and Hart (1986) and Hart and Moore (1990). Our model's technical scaffold differs substantially from this literature. We start with one defining principle of property, the right to exclude, and we trace out its implications in an economy with general interdependencies. Exclusion rights allow an agent to withhold the supply of goods, thus tilting outcomes in his favor. This monopoly dimension of property is well known; Posner and Weyl (2018) discuss its implications at length. In a production context, it often implies hold-up and inefficiency. Our analysis shows that this effect has consequences beyond a specific buyer-seller relationship,

[18] Simply introducing ownership shares is unsatisfactory since it is not obvious how share endowments (should) interact with $\omega(\cdot)$.

[19] For Definition 7(a) we could instead write that for all $i \in I$, $\mu^*(i)$ is $\succsim_i$-maximal in $\{x \in X \cup \{x_0\} | p_x \leq p_{\mu^*(i)}\}$.

the focus of standard contract-theoretic studies, due to the network structure of economic activity. Changes to the ownership structure (e.g., vertical integration) can allay these concerns (Williamson, 1971; Klein et al., 1978; Kranton and Minehart, 2000).

Despite differences with the incomplete contracts literature, our analysis is thematically related to that of Hart and Moore (1990) who develop a theory of the firm emphasizing asset control. In Hart and Moore (1990), agents make ex ante (human capital) investments and divide the surplus from production. Surplus division depends on the "control structure," which defines asset ownership and determines ex post bargaining power. Though contractable ex ante, Hart and Moore's control structure is fixed when final outcomes are determined. The endowment system $\omega(\cdot)$ in our model is similarly fixed. However, final allocations in our setting are governed by the extended endowment, $\Omega_\gamma(\cdot|\omega,\mu)$, which is endogenous to the outcome (μ,γ). The extended endowment defines each coalition's de facto power accounting for the micro-level connections in consumption and production. These details are absent from Hart and Moore's model and, we argue, are critical whenever exclusion rights govern interaction.

Production and Trading Networks Analysis of the economy's network structure dates to at least the input-output models of Leontief (1941). Carvalho and Tahbaz-Salehi (2019) survey this literature, with focus on how production networks propagate economic shocks. Our model's specifics distinguish it from this macro- and trade-oriented literature, but we share the premise that direct and indirect trading and input-output relationships transmit shocks between firms. Our solution relies on agents being able to threaten harm to (indirectly) connected parties by withholding supply (a negative shock).

The relationship between network structures and economic outcomes has been examined in the literature on trading networks and intermediation (Kranton and Minehart, 2001; Gale and Kariv, 2007; Elliott, 2015; Condorelli et al., 2017; Manea, 2018). Galeotti and Condorelli (2016) provide a survey. A consistent finding in this literature is that an agent's market power is tied to his position in the trading network. Our model echoes this intuition. If a good is a critical input for many firms, either directly or indirectly via a supply chain, its owners can block many unfavorable outcomes.

Finally, our analysis complements research by Ostrovsky (2008), Hatfield et al. (2013), and Fleiner et al. (2019). These authors extend the "matching with contracts" framework of Hatfield and Milgrom (2005) to the case of supply chains and trading networks. In such models, bilateral contracts determine the goods exchanged and the terms of trade. Our study shares this literature's motivation, but differs on both technical and conceptual grounds. The technical distinction concerns the solution concept. Stability and its generalizations are the preferred solutions in contract-based matching models. Roughly, a set of contracts is stable if there does not exist a coalition that can profitably recontract. The exclusion core, in contrast, posits that agents can veto others' assignments by invoking their direct and indirect exclusion rights. At an exclusion core outcome, no agent can profitably exercise such claims.

A conceptual contrast is also noteworthy. At a high level, the matching with contracts framework builds upon Gale and Shapley's (1962) "marriage market" model. Our model's technical roots are in Shapley and Scarf's (1974) "house exchange" economy. This difference is intriguing given the former's emphasis on *contracts* and the latter's connection to *property* (as argued above). The contracts-property dichotomy is a recognized, though fluid, distinction in legal analysis.[20] It is interesting, therefore, that two seminal models of markets have two different legal institutions in their foundations. We leave it to future research to characterize this distinction's implications for both the design and operation of markets.

A Appendix: Scalable Production

Our model assumes that a firm f produces $\varnothing$ or X_f. This framework embeds scalable production. To illustrate, suppose a firm produces two units of output, x_1 and x_2. Production of the first unit requires one input, y_1, while production of the second unit requires the *additional inputs* y_2 and y_3. Intuitively, this firm has decreasing returns to scale. To embed this situation in our model, posit there are two firms, f_1 and f_2, with production functions

$$f_1(Z) = \begin{cases} \{x_1, x^*\} & \text{if } y_1 \in Z \\ \varnothing & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(Z) = \begin{cases} x_2 & \text{if } Z \supseteq \{y_2, y_3, x^*\} \\ \varnothing & \text{otherwise} \end{cases}.$$

The first unit x_1 is produced using y_1 as an input. Production of x_1 also creates a token x^*. Production of the second unit x_2 requires inputs y_2 and y_3 and the token x^*. The latter ensures that good x_2 is produced *only if* good x_1 is also produced. (Else, x^* would not be available.) The token chains together the operation of f_1 and f_2 as if they form one entity that can scale output. The preceding construction can be repeated as required. Increasing and decreasing returns to scale production can be accommodated.

[20]Merrill and Smith (2001a, p. 774) write that "[p]roperty and contracts are bedrock institutions of the legal system, but it is often difficult to say where the one starts and the other leaves off."

B Appendix: Consumption of Multiple Goods

Our model assumes that agents have unit demand. Discrete exchange economies where an agent can consume multiple goods are ill behaved, with few positive results (Konishi et al., 2001). We can incorporate this case via a reinterpretation of "goods" and production by "firms." To sketch the argument, assume agent i prefers an apple (x_a) to a banana (x_b), but his most-preferred consumption choice is to have both an apple and a banana, $\{x_a, x_b\} \succ_i x_a \succ_i x_b$. This preference lies outside our model. Let f be an agent-specific firm that transforms an apple and a banana into an agent-specific, apple-banana composite good (x^i_{ab}) according to the production function

$$f(Z) = \begin{cases} x^i_{ab} & \text{if } Z \supseteq \{x_a, x_b\} \\ \varnothing & \text{otherwise} \end{cases}.$$

The good x^i_{ab} is essentially a relabeling of $\{x_a, x_b\}$ and, therefore, it is natural to conclude that agent i's preference would be $x^i_{ab} \succ_i x_a \succ_i x_b$. The preceding preference is within our framework. We can replicate the preceding construction for all agents as required.

C Appendix: Operation of Algorithm 1

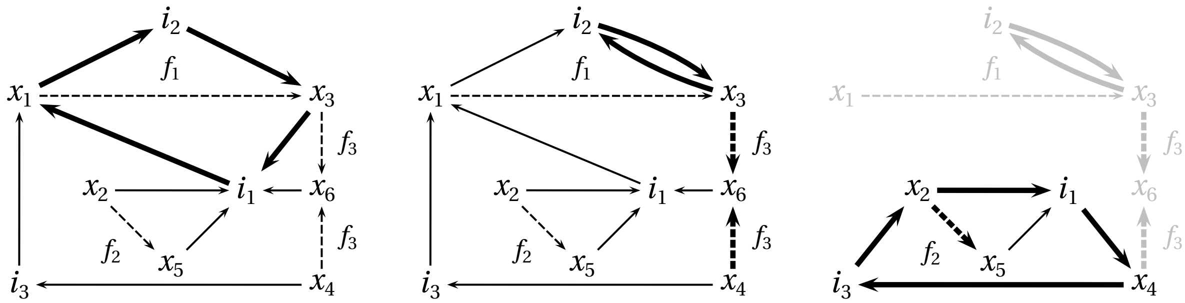
To illustrate Algorithm 1's operation, consider the following economy $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$. The set of goods is $X = \{x_1, \ldots, x_6\}$. All goods have unit capacity. Goods $\{x_1, \ldots, x_4\}$ are produced by the firms. Goods $\{x_5, x_6\}$ are primary goods. There are three firms, $F = \{f_1, f_2, f_3\}$, with production functions

$$f_1(Z) = \begin{cases} x_1 & x_3 \in Z \\ \varnothing & \text{otherwise} \end{cases}, \quad f_2(Z) = \begin{cases} x_2 & x_5 \in Z \\ \varnothing & \text{otherwise} \end{cases}, \quad \text{and} \quad f_3(Z) = \begin{cases} \{x_3, x_4\} & x_6 \in Z \\ \varnothing & \text{otherwise} \end{cases}.$$

An output of f_3 is an input for f_1. There are three agents, $I = \{i_1, i_2, i_3\}$, with preferences

$$\succ_{i_1}: x_1, x_4 \qquad \succ_{i_2}: x_3, x_2 \qquad \succ_{i_3}: x_1, x_3, x_2.$$

Each good's principals are: $C^{x_1} = \{i_2\}$, $C^{x_2} = \{i_1\}$, $C^{x_3} = \{i_1, i_2\}$, $C^{x_4} = \{i_3\}$, and $C^{x_5} = C^{x_6} = \{i_1, i_2, i_3\}$. The endowment system $\omega(\cdot)$ is given by $x \in \omega(C) \iff C^x \subseteq C$. Thus, goods x_5 and x_6 are collectively owned. Goods x_3 and x_4 have disjoint sets of principals despite being produced by the same firm.



(a) Beginning of step 1. (b) End of step 1. (c) Step 2.

Figure 8: Step-by-step operation of Algorithm 1.

Figure 8 illustrates the algorithm's step-by-step operation.

Step 1. The algorithm initializes with $I^1=\{i_1,i_2,i_3\}$, $F^1=\{f_1,f_2,f_3\}$, and $X_0^1=\{x_5,x_6\}$. The maximal, efficient (X_0^1,F^1)-feasible production network is $\gamma^1(f_1)=x_3$, $\gamma^1(f_2)=x_5$, and $\gamma^1(f_3)=x_6$. Thus, $X^1=\{x_1,\ldots,x_6\}$. The initial directed graph constructed in step 1 is illustrated in Figure 8(a). There is one cycle, but its implied allocation is infeasible. (Good x_3 is assigned to i_2 as a consumption good; it is also an input for x_1.) The cycle trimming procedure reassigns x_3 to point to i_2. This is because $x_3\in\Lambda_{\gamma^1}(x_1)$ and $x_1\to i_2$. Figure 8(b) illustrates this adjustment. The resulting cycle ($i_2\rightleftarrows x_3$) determines this step's assignments as follows:

(a) Agent i_2 is assigned x_3: $\mu(i_2)=x_3$. Thus, $\tilde{I}^1=\{i_2\}$ and $I^2=\{i_1,i_3\}$.

(b) Good x_3 must be produced by f_3. Thus, the firm is assigned its requisite input: $\gamma(f_3)=\gamma^1(f_3)=x_6$. $\tilde{F}^1=\{f_3\}$. Coincidentally, f_3 also produces x_4, which is henceforth available.

(c) The set of assigned goods with depleted capacity is $\tilde{X}^1=\{x_3,x_6\}$. The set of primary or produced goods with remaining capacity is $X_0^2=\{x_4,x_5\}$.

(d) The maximal, efficient $(X_0^2,\{f_1,f_2\})$-feasible production network is $\hat{\gamma}^1(f_1)=\varnothing$ and $\hat{\gamma}^1(f_2)=x_5$. Thus, firm f_1 cannot produce given the defined allocation. Hence, $\hat{F}^1=\{f_1\}$ and $\hat{X}^1=\{x_1\}$. Therefore, $\gamma(f_1)=\varnothing$ and $F^2=\{f_2\}$.

Step 2. At the start of step 2, $I^2=\{i_1,i_3\}$, $F^2=\{f_2\}$, and $X_0^2=\{x_4,x_5\}$. The maximal, efficient (X_0^2,F^2)-feasible production network is $\gamma^2(f_2)=x_5$ and $X^2=\{x_2,x_4,x_5\}$. Figure 8(c) presents the directed graph constructed at the beginning of step 2. There is one cycle and its implied allocation is feasible. The step's assignments are defined as follows:

(a) Agent i_1 receives x_4 ($\mu(i_1)=x_4$) and i_3 receives x_2 ($\mu(i_3)=x_2$). Thus, $\tilde{I}^2=\{i_1,i_3\}$ and $I^3=\varnothing$.

(b) Firm f_2 produces x_2 and receives its requisite input: $\gamma(f_2)=\gamma^2(f_2)=x_5$. $\tilde{F}^2=\{f_2\}$.

(c) The set of assigned goods with depleted capacity is $\tilde{X}^2 = \{x_2, x_4, x_5\}$. The set of primary or produced goods with remaining capacity is $X_0^3 = \varnothing$.

(d) As $F^2 \setminus \tilde{F}^2 = \varnothing$, assignment phase (d) is unnecessary.

Since $I^3 = \varnothing$, the algorithm terminates after step 2. The final outcome is (μ, γ) where $\mu(i_1) = x_4$, $\mu(i_2) = x_3$, $\mu(i_3) = x_2$, $\gamma(f_1) = \varnothing$, $\gamma(f_2) = x_5$, and $\gamma(f_3) = x_6$.

D Appendix: Proofs

Proof of Lemma 1. Recall that $\Omega(C|\omega,\mu) \coloneqq \omega(\bigcup_{k=0}^{\infty} C_k)$ where $C_0 = C$ and $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for each $k \geq 1$. We make two preliminary observations. First, since $C_k \subseteq C_{k+1}$ and $\omega(\cdot)$ is monotone, it follows that $\Omega(C|\omega,\mu) = \bigcup_{k=0}^{\infty} \omega(C_k)$. And second, $C_k = C \cup (\mu^{-1} \circ \omega)(C_{k-1})$ for each $k \geq 1$. We can prove this fact by induction. The base case is true since $C_1 = C_0 \cup (\mu^{-1} \circ \omega)(C_0)$ and $C_0 = C$. Let $k \geq 2$ and suppose $C_{k-1} = C \cup (\mu^{-1} \circ \omega)(C_{k-2})$. By definition, $C_k = C_{k-1} \cup (\mu^{-1} \circ \omega)(C_{k-1}) = C \cup (\mu^{-1} \circ \omega)(C_{k-2}) \cup (\mu^{-1} \circ \omega)(C_{k-1})$. Since $C_{k-2} \subseteq C_{k-1}$, $\mu^{-1}(\omega(C_{k-2})) \subseteq \mu^{-1}(\omega(C_{k-1}))$. Hence, $C_k = C \cup (\mu^{-1} \circ \omega)(C_{k-1})$.

To prove the lemma it suffices to show that $Z_k = \omega(C_k)$ for all k. If $k = 0$, then $Z_0 = \omega(C) = \omega(C_0)$. Proceeding by induction, let $k \geq 1$. If $Z_{k-1} = \omega(C_{k-1})$, then $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) = \omega(C_{k-1}) \cup \omega(C \cup \mu^{-1}(\omega(C_{k-1}))) = \omega(C_{k-1}) \cup \omega(C_k) = \omega(C_k)$. $\square$

The proof of Theorem 2 invokes the following lemma.

Lemma 2. *Let $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$ be an economy satisfying (A1)–(A4). Let (μ, γ) be a feasible outcome and $C \subseteq I$. Let $\dot{Z}_{-1} = \varnothing$ and for each $k \geq 0$, recursively define $\dot{Z}_k \coloneqq \dot{Z}_{k-1} \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1})) \cup \lambda_\gamma(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$. For each $k \geq 0$,*

$$\lambda_\gamma\big(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\big) \supseteq \alpha_\gamma\big(\dot{Z}_{k-1} \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\big). \tag{5}$$

Proof of Lemma 2. If $k = 0$, (5) becomes $\lambda_\gamma(\omega(C)) \supseteq \alpha_\gamma(\omega(C))$. This statement is true because $\lambda_\gamma(Z) \supseteq \alpha_\gamma(Z)$ for all $Z \subseteq X$. Let $k \geq 1$ and working toward a contradiction assume that

$$x \in \alpha_\gamma\big(\dot{Z}_{k-1} \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\big), \tag{6}$$

but

$$x \notin \lambda_\gamma\big(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\big). \tag{7}$$

Since $\lambda_\gamma(Z) \supseteq \alpha_\gamma(Z)$, (7) implies

$$x \notin \alpha_\gamma\left(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\right). \tag{8}$$

Together, (6) and (8) imply that there exists a nonempty $Y^x \subseteq \dot{Z}_{k-1}$ such that (a) $Y^x \cap \omega(C \cup \mu^{-1}(\dot{Z}_{k-1})) = \varnothing$ and (b) $x \in \alpha_\gamma(Y^x \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$.

If $y \in Y^x \subseteq \dot{Z}_{k-1}$, then (i) $y \in \omega(C \cup \mu^{-1}(\dot{Z}_{k'-1}))$ for some $k' < k$ and/or (ii) $y \in \lambda_\gamma(\omega(C \cup \mu^{-1}(\dot{Z}_{k'-1})))$ for some $k' < k$. Condition (i) cannot be true. By monotonicity of $\omega(\cdot)$, $\omega(C \cup \mu^{-1}(\dot{Z}_{k'-1})) \subseteq \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))$. Thus, if $y \in \omega(C \cup \mu^{-1}(\dot{Z}_{k'-1}))$, then $y \in Y^x \cap \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))$, contradicting property (a) above. Thus, (ii) holds for all $y \in Y^x$. Therefore, $Y^x \subseteq \lambda_\gamma(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$. This implies,[21] $\alpha_\gamma(Y^x \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))) \subseteq \lambda_\gamma(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$. By property (b), $x \in \alpha_\gamma(Y^x \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$. Hence, $x \in \lambda_\gamma(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})))$ contradicting (7). □

Proof of Theorem 2. Fix $\mathscr{E} = \langle I, F, X, \succ, \omega \rangle$. Suppose $\gamma \in \Gamma$ satisfies (3). Consider the exchange economy $\dot{\mathscr{E}} = \langle \dot{I}, \dot{X}, \dot{\succ}, \dot{\omega} \rangle$ derived from $\mathscr{E}$ as follows: $\dot{I} = I$; $\dot{X} = (X_0 \cup f_F(\gamma)) \cap \{x \in X \mid |\{f \in F \mid x \in \gamma(f)\}| < q_x\}$; for each i, $\dot{\succ}_i$ equals $\succ_i$ restricted to $\dot{X} \cup \{x_0\}$; and, $\dot{\omega}(C) := (\omega(C) \cup \lambda_\gamma(\omega(C))) \cap \dot{X}$ for each $C \subseteq \dot{I}$.

It is clear that $\dot{\omega}$ satisfies assumptions (A1)–(A3). To verify (A4), observe that

$$\begin{aligned}
\bigcap_{C \in \{C' \mid x \in \dot{\omega}(C')\}} C &= \left(\bigcap_{C \in \{C' \mid x \in \lambda_\gamma(\omega(C'))\}} C \right) \cap \left(\bigcap_{C \in \{C' \mid x \in \omega(C')\}} C \right) \\
&= \left(\bigcap_{C \in \{C' \mid \omega(C') \in \Lambda_\gamma(x)\}} C \right) \cap C^x \\
&\supseteq \left(\bigcap_{C \in \{C' \mid \omega(C') \in \Lambda_\gamma(x)\}} \left(\bigcup_{z \in \omega(C)} C^z \right) \right) \cap C^x && (9) \\
&\supseteq \left(\bigcap_{Z \in \Lambda_\gamma(x)} \left(\bigcup_{z \in Z} C^z \right) \right) \cap C^x && (10) \\
&= \bigcap_{Z \in \Lambda_\gamma(x)} \left(\bigcup_{z \in Z} C^z \right) \qquad (\because x \in \Lambda_\gamma(x)) \\
&\neq \varnothing.
\end{aligned}$$

Line (9) is because $C^z \subseteq C$ for all $z \in \omega(C)$ (by (A4)). And so, $\bigcup_{z \in \omega(C)} C^z \subseteq C$. Line (10) follows

[21] Recall that $\lambda_\gamma(Z) = \bigcup_{k=0}^{\infty} A_k$ where $A_0 = Z$ and $A_k = A_{k-1} \cup \alpha_\gamma(A_{k-1})$. If $Y \subseteq \lambda_\gamma(Z)$, then there exists K such that $Y \subseteq \bigcup_{k=0}^{K} A_k = A_K$. Thus, $\alpha_\gamma(Y \cup Z) \subseteq \alpha_\gamma(A_K) \subseteq A_{K+1} \subseteq \lambda_\gamma(Z)$.

from the fact that $\{Z \in \Lambda_\gamma(x) | \exists C' \subseteq I$ such that $\omega(C') = Z\} \subseteq \Lambda_\gamma(x)$. Since $\dot{\omega}$ satisfies (A1)–(A4), Theorem 1 implies that $\dot{\mathscr{E}}$ has an exclusion core allocation, μ.

Next, observe that (μ, γ) is feasible in $\mathscr{E}$. This is because $\mu\colon I \to \dot{X} \cup \{x_0\}$ only assigns goods that are available given the firms' input assignments and output. To verify that (μ, γ) is in the ex post exclusion core of $\mathscr{E}$, it suffices to show that $\Omega_\gamma(C|\omega, \mu) \cap \dot{X} \subseteq \Omega(C|\dot{\omega}, \mu)$ where $\Omega(C|\dot{\omega}, \mu)$ is defined in (1a,b) and $\Omega_\gamma(C|\omega, \mu)$ is defined in (2a,b).

First, note that $Z_0 \cap \dot{X} = \omega(C) \cap \dot{X} \subseteq (\omega(C) \cup \lambda_\gamma(\omega(C))) \cap \dot{X} = \dot{\omega}(C) = \dot{Z}_0$. Proceeding by induction, let $k \geq 1$ and suppose that $Z_{k-1} \cap \dot{X} \subseteq \dot{Z}_{k-1}$. Since, $\dot{Z}_k = \dot{Z}_{k-1} \cup \dot{\omega}(C \cup \mu^{-1}(\dot{Z}_{k-1}))$,

$$\begin{aligned}
\dot{Z}_k &= \dot{Z}_{k-1} \cup \left[\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})) \cup \lambda_\gamma\left(\omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\right)\right] \cap \dot{X} \\
&\supseteq \dot{Z}_{k-1} \cup \left[\omega(C \cup \mu^{-1}(\dot{Z}_{k-1})) \cup \alpha_\gamma\left(\dot{Z}_{k-1} \cup \omega(C \cup \mu^{-1}(\dot{Z}_{k-1}))\right)\right] \cap \dot{X} && (11) \\
&\supseteq \left[(Z_{k-1} \cap \dot{X}) \cup \omega(C \cup \mu^{-1}(Z_{k-1} \cap \dot{X})) \cup \alpha_\gamma(Z_{k-1} \cap \dot{X})\right] \cap \dot{X} && (12) \\
&= \left[Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})\right] \cap \dot{X} && (13) \\
&= Z_k \cap \dot{X}.
\end{aligned}$$

Line (11) follows from Lemma 2. Line (12) follows from the induction hypothesis and the monotonicity of ω, μ^{-1}, and α_γ. Line (13) is because $\mu^{-1}(Z_{k-1} \cap \dot{X}) = \mu^{-1}(Z_{k-1})$ and $\alpha_\gamma(Z_{k-1} \cap \dot{X}) = \alpha_\gamma(Z_{k-1})$. (Only available goods may be assigned or be part of a critical set of inputs.) Since $Z_k \cap \dot{X} \subseteq \dot{Z}_k$ for each k, $\Omega_\gamma(C|\omega, \mu) \cap \dot{X} = \bigcup_{k=0}^{\infty}(Z_k \cap \dot{X}) \subseteq \bigcup_{k=0}^{\infty} \dot{Z}_k = \Omega(C|\dot{\omega}, \mu)$. $\square$

The proof of Theorem 3 relies on several lemmas. Lemmas 3–8 are preliminaries invoked in subsequent arguments. Lemma 9 shows that Algorithm 1's output is feasible and Lemma 10 demonstrates that it is in the ex ante exclusion core of an acyclic economy. Definition 8 introduces the condensation of an economy. Each condensed economy has an ex ante exclusion core outcome (Lemma 11). Theorem 3's proof concludes by using Lemma 11 to show that every economy satisfying the theorem's hypotheses has an ex ante exclusion core outcome.

Lemma 3. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ be an economy satisfying (A1)–(A4) and (B1). Let $Z \subseteq X$, $\gamma \in \Gamma$, and consider $\lambda_\gamma(Z) = \bigcup_{k=0}^{\infty} A_k$ where $A_0 = Z$ and $A_k = A_{k-1} \cup \alpha_\gamma(A_{k-1})$ for each $k \geq 1$.*

(a) If $a_K \in A_K$ and $a_K \notin A_{K-1}$, then there exists a sequence $(a_K, a_{K-1}, \ldots, a_0)$ such that $a_k \in A_k$ for each $k \geq 0$ and $a_k \in \alpha_\gamma(a_{k-1})$ for each $k \geq 1$.

(b) For all $x \in X$, every $Z \in \Lambda_\gamma(x)$ consists of a single element.

(c) If $z \in \Lambda_\gamma(y)$ and $y \in \Lambda_\gamma(x)$, then $z \in \Lambda_\gamma(x)$.

(d) If $y \in \Lambda_\gamma(x)$, then $\bigcap_{z\in\Lambda_\gamma(x)} C^z \subseteq \bigcap_{z\in\Lambda_\gamma(y)} C^z$.

(e) If the economy's input network Φ is acyclic, $[x \neq y \ \&\ y \in \Lambda_\gamma(x)] \implies x \notin \Lambda_\gamma(y)$.

(f) Suppose F_k is a strongly connected component of the economy's input network Φ and $f, f' \in F_k$ are distinct firms. If $x \in X_f$ is produced at γ and $y \in W_{f'}$, then $y \in \Lambda_\gamma(x)$.

Proof of Lemma 3. (a) Suppose $a_K \in A_K$ and $a_K \notin A_{K-1}$. Thus, $a_K \in \alpha_\gamma(A_{K-1})$ and A_{K-1} is critical for x at γ. Since each firm's production function in Leontief, $a_K \in \alpha_\gamma(a_{K-1})$ for some $a_{K-1} \in A_{K-1}$. Moreover, $a_{K-1} \notin A_{K-2}$ (else, $a_K \in A_{K-1}$, which is assumed not true). Repeating this same construction, we can define a sequence $(a_K, a_{K-1}, \ldots, a_1, a_0)$ such that $a_k \in A_k$ for each k and $a_k \in \alpha_\gamma(a_{k-1})$.

(b) Suppose $Z \in \Lambda_\gamma(x)$. Thus, $x \in \lambda_\gamma(Z)$. It suffices to show that there exists $z \in Z$ such that $x \in \lambda_\gamma(z)$. The result is immediate if $x \in Z$. Thus, suppose $x \notin Z$. Since $x \in \lambda_\gamma(Z) = \bigcup_{k=0}^\infty A_k$, $x \in A_K$ and $x \notin A_{K-1}$ for some some $K \geq 1$. By part (a), there is a sequence $x = a_K, a_{K-1}, \ldots, a_1, a_0 = z$ such that $a_k \in \alpha_\gamma(a_{k-1})$ for each $k \geq 1$ and $z \in Z$. Given z, consider the sequence $A_0^z = \{z\}$ and $A_k^z = A_{k-1}^z \cup \alpha_\gamma(A_{k-1}^z)$ for each $k \geq 1$. Clearly, $a_k \in A_k^z$ for each $k = 0, \ldots, K$. And so, $x \in \bigcup_{k=0}^\infty A_k^z = \lambda_\gamma(z)$.

(c) If $z \in \Lambda_\gamma(y)$, then $y \in \lambda_\gamma(z) = \bigcup_{k=0}^\infty A_k^z$ where $A_0^z = \{z\}$ and $A_k^z = A_{k-1}^z \cup \alpha_\gamma(A_{k-1}^z)$. Likewise, if $y \in \Lambda_\gamma(x)$, then $x \in \lambda_\gamma(y) = \bigcup_{k=0}^\infty A_k^y$ where $A_0^y = \{y\}$ and $A_k^y = A_{k-1}^y \cup \alpha_\gamma(A_{k-1}^y)$. Let K be the smallest value for which $y \in \bigcup_{k=0}^K A_k^z$. Thus, for all $k \geq K$, $A_{k-K}^y \subseteq A_k^z$. Hence, $\bigcup_{k=0}^\infty A_k^y \subseteq \bigcup_{k=0}^\infty A_k^z$. Therefore, $x \in \lambda_\gamma(z)$ and $z \in \Lambda_\gamma(x)$.

(d) By part (c), $[z \in \Lambda_\gamma(y) \ \&\ y \in \Lambda_\gamma(x)] \implies z \in \Lambda_\gamma(x)$. Thus, $\Lambda_\gamma(y) \subseteq \Lambda_\gamma(x)$. Hence, $\bigcap_{z\in\Lambda_\gamma(x)} C^z \subseteq \bigcap_{z\in\Lambda_\gamma(y)} C^z$.

(e) Suppose $x \neq y$ and $y \in \Lambda_\gamma(x)$. Thus, $x \in \lambda_\gamma(y)$. Given part (a), there exists a sequence of goods $(a_K, \ldots, a_0)$ such that $x = a_K$, $y = a_0$ and $a_k \in \alpha_\gamma(a_{k-1})$ for all $k \geq 1$. Because production functions are Leontief, there is a link from the firm producing a_k to the firm producing a_{k-1} in Φ. Hence, there is a path in Φ from the producer of x to the producer of y. If $x \in \Lambda_\gamma(y)$, then the same reasoning implies that there exists a path in Φ from the producer of y to the producer of x. As Φ is acyclic, this is impossible. Thus, $x \notin \Lambda_\gamma(y)$.

(f) Because $f, f' \in F_k$, there is a path in Φ such that $f = f^1 \to \cdots \to f^L = f'$. If $y \in W_{f'}$, then it is critical for the production of all $x' \in X_{f^L}$. Since $f^{L-1} \to f^L$, there exist $y^L \in X_{f^L} \cap W_{f^{L-1}}$. This good is critical for the production of all $x' \in X_{f^{L-1}}$. Continuing in this way, we can construct a sequence $y^\ell \in X_{f^\ell} \cap W_{f^{\ell-1}}$ for all $\ell \geq 2$ where y^ℓ is a critical input for firm $f^{\ell-1}$'s production. Thus, $x = y^1 \in \alpha_\gamma(y^2), y^2 \in \alpha_\gamma(y^3), \ldots, y^L \in \alpha_\gamma(y)$. Thus, $x \in \lambda_\gamma(y)$ and, therefore, $y \in \Lambda_\gamma(x)$. $\square$

Lemma 4. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ be an economy satisfying (A1)–(A4) and (B1). Let (μ, γ) be a feasible outcome and $C \subseteq I$. Let $Z_{-1} = \varnothing$ and $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$ for each $k \geq 0$. If $x \in Z_k$, then there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-1}))$ such that $y \in \Lambda_\gamma(x)$.*

Proof of Lemma 4. If $x \in Z_0$, then $x \in \omega(C)$ and $x \in \Lambda_\gamma(x)$. Proceeding by induction, let $k \geq 1$ and suppose $x \in Z_{k'} \implies [\exists y \in \omega(C \cup \mu^{-1}(Z_{k'-1}))$ s.t. $y \in \Lambda_\gamma(x)]$ is true for all $k' \leq k-1$. Let $x \in Z_k$. There are three cases.

Case 1. If $x \in Z_{k-1}$, then by the induction hypothesis there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-2})) \subseteq \omega(C \cup \mu^{-1}(Z_{k-1}))$ such that $y \in \Lambda_\gamma(x)$.

Case 2. If $x \in \omega(C \cup \mu^{-1}(Z_{k-1}))$, then the conclusion follows trivially $(x = y)$.

Case 3. If $x \in \alpha_\gamma(Z_{k-1})$, then Z_{k-1} is a critical set of inputs for x. By (B1), there exists some $x' \in Z_{k-1}$ that is a critical input for x, i.e., $x' \in \Lambda_\gamma(x)$. By the induction hypothesis, there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-2})) \subseteq \omega(C \cup \mu^{-1}(Z_{k-1}))$ such that $y \in \Lambda_\gamma(x')$. Since, $x' \in \Lambda_\gamma(x)$, Lemma 3(c) implies that $y \in \Lambda_\gamma(x)$. □

Lemma 5. *Consider an economy $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ satisfying (B1) and (B2). Let $X' \subseteq X$ and $F' \subseteq F$. A maximal, efficient (X', F')-feasible production network exists and is unique.*

Proof of Lemma 5. The empty production network (i.e., $\gamma(f) = \varnothing$ for all $f \in F'$) is efficient and (X', F')-feasible. Thus, a maximal, efficient (X', F')-feasible production network exists. To show uniqueness, let $\gamma \neq \gamma'$ be maximal, efficient (X', F')-feasible production networks. Let $\hat{\gamma}(f) = \gamma(f) \cup \gamma'(f)$ for all $f \in F'$. The network $\hat{\gamma}$ is efficient because γ and γ' are efficient and each firm has a Leontief production function. (A producing firm must be assigned a unique set of inputs; a non-producing firm must be assigned no inputs.) Since $\hat{\gamma} \supseteq \gamma$ and $\hat{\gamma} \neq \gamma$, $\hat{\gamma}$ must not be (X', F')-feasible. Since γ and γ' are (X', F')-feasible, $\hat{\gamma}(F') \subseteq X' \cup f_{F'}(\hat{\gamma})$. Therefore, there exists a good x such that $|\{f \in F' | x \in \hat{\gamma}(f)\}| > q_x$. Since each production function is Leontief, there are strictly more than q_x firms in F' that require x as an input. At least one of these firms cannot produce at $\overline{\gamma}$, contradicting (B2). □

Lemma 6. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ be an economy satisfying (A1)–(A4) and (B1)–(B3). Let $X' \subseteq X$ and $F' \subseteq F$. If γ is an (X', F')-feasible production network, then $C(x|\gamma) = \bigcap_{z \in \Lambda_\gamma(x)} C^z \neq \varnothing$ for all $x \in X' \cup f_{F'}(\gamma)$.*

Proof of Lemma 6. By Lemma 3(b), $C(x|\gamma) = \bigcap_{z \in \Lambda_\gamma(x)} C^z$ for all γ. By (B3), $C(x|\overline{\gamma}) = \bigcap_{z \in \Lambda_{\overline{\gamma}}(x)} C^z \neq \varnothing$. Therefore, to prove the lemma it suffices to show that $\Lambda_\gamma(x) \subseteq \Lambda_{\overline{\gamma}}(x)$ for all $x \in X' \cup f_{F'}(\gamma)$.

Consider the (X', F')-feasible production network γ. Let $x \in X' \cup f_{F'}(\gamma)$. By Lemma 3(b), $\Lambda_\gamma(x)$ contains only singletons. Let $z \in \Lambda_\gamma(x)$. Therefore, $x \in \lambda_\gamma(z) = \bigcup_{k=0}^{\infty} A_k^\gamma$ where $A_0^\gamma = \{z\}$ and $A_k^\gamma = A_{k-1}^\gamma \cup \alpha_\gamma(A_{k-1}^\gamma)$. (We include the γ superscript on A_k^γ for clarity.) By Lemma 3(a), there exists a sequence $(a_K, a_{K-1}, \ldots, a_0)$ such that $x = a_K$, $a_0 = z$, and $a_k = \alpha_\gamma(a_{k-1})$ for each $k \geq 1$. Since production functions are Leontief, if a_{k-1} is critical for a_k at γ, it must be critical for a_k at $\overline{\gamma}$. Thus, $a_k = \alpha_{\overline{\gamma}}(a_{k-1})$ for each $k \geq 1$. Hence, $a_0 \in A_0^{\overline{\gamma}}$ and $a_k \in A_k^{\overline{\gamma}} = A_{k-1}^{\overline{\gamma}} \cup \alpha_{\overline{\gamma}}(A_{k-1}^{\overline{\gamma}})$ for each $k \geq 1$. And so, $x \in \bigcup_{k=0}^{\infty} A_k^{\overline{\gamma}} = \lambda_{\overline{\gamma}}(z)$, which implies $z \in \Lambda_{\overline{\gamma}}(x)$. Thus, $\Lambda_\gamma(x) \subseteq \Lambda_{\overline{\gamma}}(x)$. $\square$

Lemma 7. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying (A1)–(A4) and (B1)–(B3). Consider step t of Algorithm 1 where γ^t is the maximal, efficient (X_0^t, F^t)-feasible production network and $C(x) = C(x|\gamma^t) \cap I^t$. Let $K_1 \supseteq \cdots \supseteq K_L$ be the sequence of cycles identified by iterating the "cycle trimming" procedure within this step of the algorithm. Let $x, y \in K_\ell \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_ℓ. Suppose the procedure reassigns x to point to j at iteration ℓ.*

(a) If $j \in C(y)$, then $j \in C(x)$.

(b) If $j \notin C(y)$, then some $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ was assigned by the algorithm in some step $t' < t$.

Proof of Lemma 7. Recall that $C(x) = C(x|\gamma^t) \cap I^t = (\bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z) \cap I^t$. The term in parenthesis is not empty by Lemma 6. The proof is by induction. Consider cycle K_1. Let $x, y \in K_1 \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_1. Suppose the procedure reassigns x to point to j. Since $x \in \Lambda_{\gamma^t}(y)$, $\bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z \subseteq \bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z$ by Lemma 3(d). Thus, if $j \in C(y)$, then $j \in C(x)$. Otherwise, if $j \notin C(y)$, then $C(y) = \emptyset$. Thus, all $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ must have been assigned prior to step t.

Proceeding by induction, suppose statements (a) and (b) of the lemma are true for all cycles $K_1, \ldots, K_{\ell-1}$. Let $x, y \in K_\ell \cap X$, $x \in \Lambda_{\gamma^t}(y)$, and suppose $x \to i$ and $y \to j$ within cycle K_ℓ. Suppose the procedure reassigns x to point to j. If $j \in C(y)$, then $j \in C(x)$ as above. Otherwise, $j \notin C(y)$ and there are two cases.

Case 1. $C(y) = \emptyset$. Thus, all $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ have been assigned prior to step t.

Case 2. $C(y) \neq \emptyset$. Because y is not pointing to an element of $C(y)$, it must have been reassigned to point to j in some prior iteration $\ell' < \ell$ of the cycle trimming procedure. Suppose at iteration ℓ', $x' \to j$ and $y \in \Lambda_{\gamma^t}(x')$. Invoking the induction hypothesis, if $j \in C(x')$, then $j \in C(y)$, a contradiction. Thus, $j \notin C(x')$ and there exists an $i' \in \bigcap_{z \in \Lambda_{\gamma^t}(x')} C^z \subseteq \bigcap_{z \in \Lambda_{\gamma^t}(y)} C^z$ who was assigned by the algorithm in step $t' < t$. $\square$

Lemma 8. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying (A1)–(A4) and (B1)–(B3). Let (μ, γ) be a feasible outcome identified by Algorithm 1. Suppose agent j was assigned good y at step t of Algorithm 1. For every $x \in \Lambda_\gamma(y)$, there exists $i \in C^x$ who was assigned his consumption allocation in step t or earlier.*

Proof of Lemma 8. Let $x \in \Lambda_\gamma(y)$. Because good y was assigned in step t, good x must be produced in step $s \le t$. (If x is a primary good, $s = 1$.) Let K be the final cycle identified by Algorithm 1 in step s that determines the consumption assignments of agents. There exists a good $x' \in K \cap X$ such that $x \in \Lambda_{\gamma^s}(x')$. (If $x \in K$, then $x = x'$.) Let $i' \in K \cap I^s$ be the agent in the cycle such that $x' \to i'$. Agent i' was assigned $\mu(i')$ in this step of the algorithm. Recalling that $C(x) = C(x|\gamma^t) \cap I^t = (\bigcap_{z \in \Lambda_{\gamma^t}(x)} C^z) \cap I^t$, there are three possibilities.

Case 1. $i' \in C(x')$. Because $x \in \Lambda_{\gamma^s}(x')$, $i' \in C(x') = (\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z) \cap I^s \subseteq C^x$. Thus, agent $i' \in C^x$ was assigned $\mu(i')$ in step $s \le t$.

Case 2. $i' \notin C(x')$ and $C(x') = \varnothing$. This implies that $(\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z) \cap I^s = \varnothing$. By Lemma 6, $\bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \neq \varnothing$. Thus, every $j' \in \bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \subseteq C^x$ must have been assigned $\mu(j')$ before step $s \le t$.

Case 3. $i' \notin C(x')$ and $C(x') \neq \varnothing$. If x' was pointing to an agent not in $C(x')$, it is because during the trimming procedure in step s it was reassigned to point to agent i'. By Lemma 7(b), there exists a $j' \in \bigcap_{z \in \Lambda_{\gamma^s}(x')} C^z \subseteq C^x$ who was assigned $\mu(j')$ in step $s' < s \le t$ of Algorithm 1.

In each case, there is a member of C^x assigned in or before step t. $\square$

Lemma 9. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an acyclic economy satisfying (A1)–(A4) and (B1). The outcome (μ, γ) identified by Algorithm 1 is feasible.*

Proof of Lemma 9. It suffices to show that if a good is assigned to agent i in step t of Algorithm 1, then it is not simultaneously assigned to any firm.[22] Assume the contrary. Suppose y is assigned to an agent and to a firm in step t. If y is a produced good, then it cannot be assigned to the firm that produces y (γ^t is efficient and y is not a necessary input for itself). Thus, there exists $x \neq y$ such that $y \in \Lambda_{\gamma^t}(x)$. If good x is assigned to an agent at step t, then both x and y belong to the cycle determining the consumption assignment at step t. However, this violates the trimming procedure's stopping criterion. Therefore, x is not assigned to another agent at

[22]The other possibilities are ruled out by the definition of Algorithm 1 and the feasibly of γ^t.

step t. But this implies $x \in \Lambda_{\gamma^t}(x')$ for some x' that is assigned to an agent at step t. By Lemma 3(b) and (c), $y \in \Lambda_{\gamma^t}(x')$. Thus, if $x' \neq y$, we arrive at a contradiction, as above. If $x' = y$, then we contradict Lemma 3(e), which invokes acyclicity, because $x \neq y$, $y \in \Lambda_{\gamma^t}(x)$, and $x \in \Lambda_{\gamma^t}(y)$. As each case leads to a contradiction, we conclude that no good is assigned to an agent and to a firm in step t. □

Lemma 10. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega\rangle$ be an acyclic economy satisfying (A1)–(A4) and (B1)–(B3). There exists an ex ante exclusion core outcome in $\mathscr{E}$.*

Proof of Lemma 10. Let (μ, γ) be an outcome identified by Algorithm 1 in $\mathscr{E}$. We will verify that (μ, γ) is an ex ante exclusion core outcome. To derive a contradiction, assume the contrary. Thus, there exists a feasible outcome (σ, ψ) and nonempty coalition $C \subseteq I$ such that $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and

$$\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega, \mu). \tag{14}$$

Without loss of generality, we may assume that C contains all agents for whom $\sigma(i) \succ_i \mu(i)$ is true. Algorithm 1 assures that $\sigma(i) \succ_i \mu(i) \succeq_i x_0$ for all $i \in C$.

Algorithm 1 constructed (μ, γ) sequentially by removing sets of agents $(\tilde{I}^1, \tilde{I}^2, \ldots)$ and goods $(\tilde{X}^1 \cup \hat{X}^1, \tilde{X}^2 \cup \hat{X}^2, \ldots)$. Each $i \in \tilde{I}^t$ was *assigned (his consumption allocation) in step t* and each $x \in \tilde{X}^t \cup \hat{X}^t$ was *removed from the market in step t*. The latter can occur for two reasons: (a) each $x \in \tilde{X}^t$ was assigned to an agent or to a firm in step t and its capacity was depleted; or, (b) the production of each $x \in \hat{X}^t$ became impossible given the assignments in step t. Colloquially, the firm (potentially) producing x was "shut down."

For every $i \in C$ assigned in step t (i.e., $i \in \tilde{I}^t$), $\sigma(i)$ must have been removed from the market in step $t' < t$ (i.e., $\sigma(i) \in \tilde{X}^{t'} \cup \hat{X}^{t'}$). If this were not true, then agent i would not have been pointing to his most preferred good among those available at the beginning of step t.

Claim 1. *There exists an agent j such that $\mu(j) \succ_j \sigma(j)$. Moreover, this agent was assigned his consumption allocation before any member of coalition C.*

Proof of Claim 1. Consider the good $x \in \sigma(C)$ removed from the market earliest by Algorithm 1. (If there are multiple such goods, pick any of them.) Suppose this occurs in step t. Since x was removed from the market, $x \notin \mu(C)$. (Otherwise, there would be some $x' \in \sigma(C)$ removed from the market strictly earlier than x.) Thus, $C \cap \tilde{I}^{t'} = \varnothing$ for all $t' \leq t$.

Since the assignment of x is different at (μ, γ) than at (σ, ψ), three cases are possible.

Case 1. There exists $j \in I$ such that $\mu(j) = x$. Thus, j is assigned x in step t of Algorithm 1. Since preferences are strict, $j \notin C$ and $\sigma(j) \neq \mu(j)$ imply that $\mu(j) \succ_j \sigma(j)$.

Case 2. There exists $f \in F$ such that $x \in \gamma(f)$. Thus, there is some produced good x' such that (i) x' is assigned to some agent j in step t, and (ii) $x \in \Lambda_{\gamma^t}(x')$. Since x must have capacity one, if $x \in \sigma(C)$ then it is unavailable as an input at (σ, ψ). Because each production function is Leontief, if x is unavailable as an input for f, then f cannot produce. (The input assignment at γ was efficient.) Therefore, good x' also cannot be produced at (σ, ψ). Thus, agent j's consumption must be different at (σ, ψ), i.e., $x' = \mu(j) \neq \sigma(j)$. Since $j \notin C$, it follows that $\mu(j) \succ_j \sigma(j)$.

Case 3. The firm producing good x is "shut down" at step t of Algorithm 1. This occurs only if an (indirect) input x' for the production of x becomes unavailable at step t. An indirect input becomes unavailable only if it is assigned as a consumption good to some agent j (i.e., $\mu(j) = x'$) in step t. (This is because x is available at the beginning of step t and the production network γ^t is feasible.) But, if x is consumed at (σ, ψ), then x' cannot be consumed by j at σ. Hence, $\sigma(j) \neq \mu(j)$. Since no member of C is assigned in step t or earlier, $j \notin C$ and we conclude that $\mu(j) \succ_j \sigma(j)$.

In each case, there is an agent j assigned in step t of Algorithm 1 for whom $\mu(j) \succ_j \sigma(j)$. $\diamond$

Given Claim 1, let j be the agent who was assigned earliest by Algorithm 1 and for whom $\mu(j) \succ_j \sigma(j)$. (If there are multiple such agents, choose any of them.) Let t^* be the step in which j's assignment was set. By Claim 1, each $i \in C$ was assigned strictly after step t^*.

Next, we show that $\mu(j) \notin \Omega_\gamma(C|\omega, \mu)$, which will contradict (14) and thus prove the theorem. Define $Z_\ell = \varnothing$ for all $\ell \leq -1$ and $Z_\ell = Z_{\ell-1} \cup \omega(C \cup \mu^{-1}(Z_{\ell-1})) \cup \alpha_\gamma(Z_{\ell-1})$ for each $\ell \geq 0$. Suppose $\mu(j) \in Z_0 = \omega(C)$. By Lemma 8, there exists $i \in C^{\mu(j)} \subseteq C$ who was assigned at step t^* or earlier. However, from above we know that no member of C was assigned in step t^*, or earlier, of Algorithm 1—a contradiction.

Proceeding by induction, let $k \geq 1$ and assume that for $k' = k-1$, (a) no agent in $C \cup \mu^{-1}(Z_{k'-1})$ was assigned at any step $t \leq t^*$ by Algorithm 1, and (b) $\mu(j) \notin \bigcup_{\ell=0}^{k'} Z_\ell$. We will verify that (a) and (b) are true for $k' = k$.

Verification of (a). Suppose $i \in C \cup \mu^{-1}(Z_{k-1})$ was assigned at step $t \leq t^*$ by Algorithm 1. Since $Z_0 \subseteq \cdots \subseteq Z_{k-1}$ and no member of $C \cup \mu^{-1}(Z_{k-2})$ was assigned at any step $t \leq t^*$, $i \in \mu^{-1}(Z_{k-1} \setminus Z_{k-2})$. Because $Z_{k-1} = Z_{k-2} \cup \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$, it must be the case that $\mu(i) \in \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$. There are two cases.

Case 1. $\mu(i) \in \omega(C \cup \mu^{-1}(Z_{k-2}))$. By Lemma 8, there exists $i' \in C^{\mu(i)} \subseteq C \cup \mu^{-1}(Z_{k-2})$ who was assigned at step $t \leq t^*$ of Algorithm 1—a contradiction with point (a) above.

Case 2. $\mu(i) \in \alpha_\gamma(Z_{k-2})$. Since Z_{k-2} is a critical input set for $\mu(i)$, there exists $x \in Z_{k-2}$ such that $x \in \Lambda_\gamma(\mu(i))$. By Lemma 8, there exists $i' \in C^x$ who was assigned his consumption allocation in step $t \leq t^*$ of Algorithm 1. Since i' is assigned before step t^*, by the induction hypothesis $i' \notin C \cup \mu^{-1}(Z_{k-2})$. Therefore, $x \notin \omega(C \cup \mu^{-1}(Z_{k-2}))$. By Lemma 4, there exists $y \in \omega(C \cup \mu^{-1}(Z_{k-3}))$ such that $y \in \Lambda_\gamma(x)$. By Lemma 3(c) , $y \in \Lambda_\gamma(\mu(i))$. By Lemma 8 , there exists an $i'' \in C^y \subseteq C \cup \mu^{-1}(Z_{k-3})$ who was assigned his consumption allocation in step $t \leq t^*$ of Algorithm 1. However, this is a contradiction as no member of $C \cup \mu^{-1}(Z_{k-3})$ can be assigned in step t^* or earlier.

Each case leads to a contradiction. Therefore, no agent in $C \cup \mu^{-1}(Z_{k-1})$ is assigned at step t^*, or earlier, by Algorithm 1.

Verification of (b). Toward a contradiction, suppose $\mu(j) \in \bigcup_{\ell=0}^{k} Z_\ell$. Necessarily, this implies $\mu(j) \in Z_k \setminus Z_{k-1}$ and, in particular, $\mu(j) \in \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$. Applying the same arguments (with all indices shifted up by one) from the verification of (a) above, together with the induction conclusion of (a), we reach a contradiction and establish that $\mu(j) \notin \bigcup_{\ell=0}^{k} Z_\ell$.

As the number of goods is finite, $\bigcup_{\ell=0}^{\infty} Z_\ell = \bigcup_{\ell=0}^{L} Z_\ell$ for some $L \in \mathbb{N}$. Thus, the preceding induction argument confirms that $\mu(j) \notin \bigcup_{\ell=0}^{\infty} Z_\ell$. □

Definition 8. Let $\mathcal{E} = \langle I, X, F, \succ, \omega\rangle$ be an economy satisfying (A1)–(A4) and (B1)–(B3). Let $\{F_1, \dots, F_L\}$ denote the strongly connected components of $\mathcal{E}$'s input network and for each $k = 1, \dots, L$, let $W_{\hat{f}_k} := (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ and $X_{\hat{f}_k} := (\bigcup_{f \in F_k} X_f) \cap \{x \in X \| \{f \in F_k | x \in W_f\}| < q_x\}$. The *condensation* of $\mathcal{E}$ is the economy $\hat{\mathcal{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega}\rangle$ where:

- The set of agents is $\hat{I} := I$.
- The set of firms is $\hat{F} := \{\hat{f}_1, \dots, \hat{f}_K\}$ where, relabeling if necessary, $\hat{F}$ consists of all $\hat{f}_k$ such that $W_{\hat{f}_k} \neq \varnothing$. Each firm's production function is $\hat{f}_k(Z) = X_{\hat{f}_k}$ if $Z \supseteq W_{\hat{f}_k}$ and $\hat{f}_k(Z) = \varnothing$ otherwise. Let $\tilde{F} = \{\hat{f}_{K+1}, \dots, \hat{f}_L\}$ be the set of all remaining $\hat{f}_k$ for whom $W_{\hat{f}_k} = \varnothing$.
- The set of goods is $\hat{X} := X_0 \cup (\bigcup_{\hat{f} \in \tilde{F}} X_{\hat{f}}) \cup (\bigcup_{\hat{f} \in \hat{F}} X_{\hat{f}})$. The set of primary goods is $\hat{X}_0 := X_0 \cup (\bigcup_{\hat{f} \in \tilde{F}} X_{\hat{f}})$.
- The preference of each $i \in \hat{I}$ equals $\succ_i$ restricted to $\hat{X}$, i.e., $\hat{\succ}_i = \succ_i|_{\hat{X}}$.
- The endowment of each coalition $C \subseteq \hat{I}$ is defined as $x \in \hat{\omega}(C) \iff [x \in \hat{X} \;\&\; \hat{C}^x \subseteq C]$ and each $\hat{C}^x$ is defined as follows. If $x \in X_0$, then $\hat{C}^x := C^x$. Otherwise, $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F} \cup \tilde{F}$ and
$$\hat{C}^x := C^x \cap \left(\bigcap\nolimits_{y \in Y_{\hat{f}_k}} C^y\right) \tag{15}$$

where $Y_{\hat{f}_k} = (\bigcup_{f \in F_k} W_f) \cap (\bigcup_{f \in F_k} X_f)$.

Lemma 11. *Let $\mathscr{E} = \langle I, X, F, \succ, \omega \rangle$ be an economy satisfying (A1)–(A4) and (B1)–(B3). There exists an ex ante exclusion core outcome in the condensation of $\mathscr{E}$.*

Proof of Lemma 11. It suffices to show that the condensation $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega} \rangle$ satisfies the hypotheses of Lemma 10. First, $\hat{\mathscr{E}}$ is acyclic. This is because the input network of $\hat{\mathscr{E}}$ is a subgraph of the condensation of the input network of $\mathscr{E}$. The condensation of a directed graph is acyclic (Bondy and Murty, 2008, pp. 91–92). Second, by definition, each $\hat{f}_k \in \hat{F}$ has a Leontief production function that is monotone and satisfies the no free lunch property. We demonstrate the remaining three requirements as separate claims.

Claim 1. *The endowment system $\hat{\omega}$ satisfies (A1)–(A4).*

Proof of Claim 1. It suffices to show that $\hat{C}^x \neq \varnothing$ for each $x \in \hat{X}$. When this is true, properties (A1)–(A4) follow from the definition of $\hat{\omega}$. If $x \in X_0$, then $\hat{C}^x = C^x \neq \varnothing$. If $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F} \cup \tilde{F}$, then $\hat{C}^x = C^x \cap (\bigcap_{y \in Y_{\hat{f}_k}} C^y)$ where $Y_{\hat{f}_k} = (\bigcup_{f \in F_k} W_f) \cap (\bigcup_{f \in F_k} X_f)$. If $y \in Y_{\hat{f}_k}$, then $y \in \Lambda_{\overline{\gamma}}(x)$ by Lemma 3(f). By Lemma 3(b), $\bigcap_{Z \in \Lambda_{\overline{\gamma}(x)}} (\bigcup_{z \in Z} C^z) \neq \varnothing$ reduces to $\bigcap_{z \in \Lambda_{\overline{\gamma}(x)}} C^z \neq \varnothing$. Thus,

$$\hat{C}^x = C^x \cap \left(\bigcap_{y \in Y_{\hat{f}_k}} C^y \right) \supseteq C^x \cap \left(\bigcap_{y \in \Lambda_{\overline{\gamma}}(x)} C^y \right) = \bigcap_{z \in \Lambda_{\overline{\gamma}}(x)} C^z \neq \varnothing \tag{16}$$

where the second equality follows from the fact that $x \in \Lambda_{\overline{\gamma}}(x)$. ⋄

Claim 2. *There exists a feasible production network $\gamma' \colon \hat{F} \to 2^{\hat{X}}$ such that $\hat{X} = \hat{X}_0 \cup \hat{f}_{\hat{F}}(\gamma')$.*

Proof of Claim 2. Define $\gamma' \colon \hat{F} \to 2^{\hat{X}}$ as follows. Let $\gamma'(\hat{f}_k) = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ for each $\hat{f}_k \in \hat{F}$. At γ', the output of $\hat{f} \in \hat{F}$ is $X_{\hat{f}}$. Thus, $\hat{X}_0 \cup \hat{f}_{\hat{F}}(\gamma') = \hat{X}_0 \cup (\bigcup_{\hat{f} \in \hat{F}} X_{\hat{f}}) = \hat{X}$.

Next we verify that γ' is feasible. Suppose $x \in \gamma'(\hat{F})$. Thus, $x \in \gamma'(\hat{f}_k) = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ for some $\hat{f}_k \in \hat{F}$. In particular, $x \in W_f$ for some $f \in F_k$. There are two possibilities. If x is a primary good, then $x \in X_0 \subseteq \hat{X}_0$. Otherwise, $x \in X_{f'}$ for some $f' \in F_\ell \neq F_k$. We know that $X_{\hat{f}_\ell} = (\bigcup_{f \in F_\ell} X_f) \cap \{x \in X | q_x > |\{f \in F_\ell | x \in W_f\}|\}$. Clearly, $x \in X_{\hat{f}_\ell}$ if and only if (strictly) fewer than q_x other firms in F_ℓ employ x as an input. If $q_x = \infty$, this is trivially true. If $q_x = 1$ and $f'' \in F_\ell$ uses x as an input, then f and f'' cannot both produce in $\mathscr{E}$ at $\overline{\gamma}$, contradicting (B2). Thus, $\gamma'(\hat{F}) \subseteq \hat{X}_0 \cup \hat{f}_{\hat{F}}(\gamma')$.

Finally, suppose $q_x < |\{\hat{f} \in \hat{F} | x \in \gamma'(\hat{f})\}|$ for some $x \in \hat{X}$. Since $q_x = 1$, there exist two or more firms that are assigned x as an input at γ'. This implies there exist $f, f' \in F$ such that $W_f \cap W_{f'} \neq \varnothing$. Thus, both f and f' cannot produce at $\overline{\gamma}$, contradicting (B2). ⋄

Claim 3. $\hat{C}(x|\gamma') \coloneqq \bigcap_{Z \in \Lambda_{\gamma'}(x)} (\bigcup_{z \in Z} \hat{C}^z) \neq \varnothing$ *for all* $x \in \hat{X}$.

Proof of Claim 3. By Lemma 3(b), $\hat{C}(x|\gamma') = \bigcap_{Z\in\Lambda_{\gamma'}(x)}(\bigcup_{z\in Z}\hat{C}^z) = \bigcap_{z\in\Lambda_{\gamma'}(x)}\hat{C}^z$. If x is a primary good or is not produced at γ', then $\Lambda_{\gamma'}(x) = \{x\}$ and $\hat{C}(x|\gamma') = \hat{C}^x \neq \varnothing$. Otherwise, $x \in X_{\hat{f}_k}$ for some $\hat{f}_k \in \hat{F}$. If z is an (indirect) critical input for x at γ', it must also be an (indirect) critical input for x at $\overline{\gamma}$ in $\mathscr{E}$. This is because each firm's production function is Leontief. Therefore, $\Lambda_{\gamma'}(x) \subseteq \Lambda_{\overline{\gamma}}(x)$. Thus, $\bigcap_{z\in\Lambda_{\gamma'}(x)}\hat{C}^z \supseteq \bigcap_{z\in\Lambda_{\gamma'}(x)}(\bigcap_{y\in\Lambda_{\overline{\gamma}}(z)}C^y) \supseteq \bigcap_{y\in\Lambda_{\overline{\gamma}}(x)}C^y \neq \varnothing$. The first set inclusion is by (16). The second is because if z is an indirect critical input x at γ' and y is an indirect critical input for z at $\overline{\gamma}$, then y is an indirect critical input for x at $\overline{\gamma}$. The final inequality is true because (B3) holds in $\mathscr{E}$ and by Lemma 3(b). $\diamond$

$\square$

Proof of Theorem 3. Consider the economy $\mathscr{E} = \langle I, F, X, \succ, \omega\rangle$. Let $\{F_1, \ldots, F_L\}$ be the strongly connected components of its input network. By Lemma 11, its condensation $\hat{\mathscr{E}} = \langle \hat{I}, \hat{F}, \hat{X}, \hat{\succ}, \hat{\omega}\rangle$ has an exclusion core outcome $(\hat{\mu}, \hat{\gamma})$. As in Definition 8, each $\hat{f}_k \in \hat{F} \cup \tilde{F}$ is defined with respect to the corresponding strongly connected component F_k. If $\hat{f}_k \in \hat{F}$, then $W_{\hat{f}_k} \neq \varnothing$. If $\hat{f}_k \in \tilde{F}$, then $W_{\hat{f}_k} = \varnothing$ and $X_{\hat{f}_k} \subseteq \hat{X}_0$ in $\hat{\mathscr{E}}$. Define the outcome (μ, γ) in $\mathscr{E}$ as follows. For each $i \in I$, let $\mu(i) = \hat{\mu}(i)$. For each $f \in F_k \subseteq F$, let

$$\gamma(f) = \begin{cases} W_f & \text{if } \hat{f}_k \in \hat{F} \text{ produces its output } X_{\hat{f}_k} \text{ at } (\hat{\mu}, \hat{\gamma}) \text{ in } \hat{\mathscr{E}} \text{ (i.e., } \hat{\gamma}(\hat{f}_k) \neq \varnothing) \\ W_f & \text{if } \hat{f}_k \in \tilde{F} \\ \varnothing & \text{otherwise} \end{cases}.$$

We will verify that (μ, γ) is an exclusion core outcome in $\mathscr{E}$.

Claim 1. *The outcome* (μ, γ) *is feasible in* $\mathscr{E}$.

Proof of Claim 1. Let $x \in \mu(I) \cup \gamma(F)$. Suppose $x \notin X_0 \cup \{x_0\}$. There are two cases.

Case 1. If $x \in \mu(I)$, then x is available at $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. Since $x \notin X_0 \cup \{x_0\}$, there exists some $\hat{f}_k \in \hat{F} \cup \tilde{F}$ such that $x \in X_{\hat{f}_k}$ and some $f \in F_k$ such that $x \in X_f$. By definition of $\gamma(\cdot)$, $\gamma(f) = W_f$. Thus, x is produced at (μ, γ) in $\mathscr{E}$, i.e., $x \in f_F(\gamma)$.

Case 2. Suppose $x \in \gamma(F)$. Thus, there exists $f \in F_k \subseteq F$ such that $x \in W_f = \gamma(f)$. Since $\gamma(f) \neq \varnothing$, $X_{\hat{f}_k}$ is available at $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. There are two subcases.

(a) If $x \in W_{\hat{f}_k}$, then $x \in X_{\hat{f}_\ell}$ for some $\hat{f}_\ell \in \hat{F} \cup \tilde{F}$ and $\hat{f}_\ell \neq \hat{f}_k$. This implies there exists some $f' \in F_\ell \subseteq F$ such that $x \in X_{f'}$ and, by definition of $\gamma(\cdot)$, $\gamma(f') = W_{f'}$. Therefore, x is produced at (μ, γ) in $\mathscr{E}$, i.e., $x \in f_F(\gamma)$.

(b) If $x \notin W_{\hat{f}_k}$, then $\hat{f}_k \in \tilde{F}$ and the producer of x, say f', must belong to the same strongly connected component F_k. In this case, the definition of $\gamma(\cdot)$ implies that $\gamma(f') = W_{f'}$. Thus, good x is produced by f' at (μ, γ) in $\mathscr{E}$, i.e., $x \in f_F(\gamma)$.

Cases 1 and 2 imply that if $x \notin X_0 \cup \{x_0\}$, then $x \in f_F(\gamma)$. Hence, $\mu(I) \cup \gamma(F) \subseteq X_0 \cup \{x_0\} \cup f_F(\gamma)$.

Next, consider good x with capacity $q_x = 1$. The following three points together imply that $|\{i \in I | \mu(i) = x\}| + |\{f \in F | x \in \gamma(f)\}| \leq q_x$ for all $x \in X$ and prove the claim.

(i) At most one agent can be assigned x at (μ, γ) in $\mathscr{E}$. This is because $\mu(i) = \hat{\mu}(i)$ for each i and $(\hat{\mu}, \hat{\gamma})$ is feasible in $\hat{\mathscr{E}}$.

(ii) At most one firm can be assigned x at (μ, γ) in $\mathscr{E}$. To see this, suppose $f \neq f'$ are both assigned x as an input. Because $q_x = 1$, firms f and f' cannot both produce at $\overline{\gamma}$, a contradiction since all goods are produced at $\overline{\gamma}$.

(iii) It is impossible for an agent and a firm to be simultaneously assigned x at (μ, γ) in $\mathscr{E}$. Suppose this was not true and $\mu(i) = x$ and $x \in \gamma(f) = W_f$. By definition, $\hat{\mu}(i) = x$ in $\hat{\mathscr{E}}$. Since firm $f \in F_k \subseteq F$ is assigned an input at (μ, γ), the set of goods $X_{\hat{f}_k}$ must be available at $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. If $x \in W_{\hat{f}_k} = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ then we have a contradiction as x would be simultaneously assigned to i and $\hat{f}_k$ in the (feasible) outcome $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$. Thus, $x \in \bigcup_{f \in F_k} X_f$ and, because x is assigned to agent i at $(\hat{\mu}, \hat{\gamma})$, $x \in X_{\hat{f}_k} = (\bigcup_{f \in F_k} X_f) \cap \{x \in X || \{f \in F_k | x \in W_f\}| < q_x\}$. But then $|\{f \in F_k | x \in W_f\}| < q_x = 1$, which is a contradiction since the set F_k contains at least one firm that uses x as an input. $\diamond$

Next we show that (μ, γ) cannot be ex ante exclusion blocked in $\mathscr{E}$. Suppose the contrary. Thus, there exists a coalition $C \subseteq I$ and a feasible outcome (σ, ψ) in $\mathscr{E}$ such that $\sigma(i) \succ_i \mu(i)$ for all $i \in C$ and $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C | \omega, \mu)$. Without loss of generality, we may assume that C contains all agents i for whom $\sigma(i) \succ_i \mu(i)$ is true and that ψ is efficient.

Consider the outcome $(\hat{\sigma}, \hat{\psi})$ in $\hat{\mathscr{E}}$ defined as follows. For each $i \in \hat{I}$, $\hat{\sigma}(i) = \sigma(i)$. For each $\hat{f}_k \in \hat{F}$, $\hat{\psi}(\hat{f}_k) = (\bigcup_{f \in F_k} \psi(f)) \setminus (\bigcup_{f \in F_k} X_f)$. We will show that coalition C can ex ante exclusion block the outcome $(\hat{\mu}, \hat{\gamma})$ in $\hat{\mathscr{E}}$ with $(\hat{\sigma}, \hat{\psi})$. This will contradict $(\hat{\mu}, \hat{\gamma})$ being in the ex ante exclusion core of $\hat{\mathscr{E}}$, thus proving the theorem. The argument's remainder is divided into three claims implying this conclusion.

Claim 2. *The outcome $(\hat{\sigma}, \hat{\psi})$ is feasible in $\hat{\mathscr{E}}$.*

Proof of Claim 2. Let $x \in \hat{\sigma}(\hat{I}) \cup \hat{\psi}(\hat{F})$. Suppose $x \notin X_0 \cup \{x_0\}$. There are two cases.

Case 1. If $x \in \hat{\sigma}(\hat{I})$, then $\hat{\sigma}(i) = x$ for some $i \in \hat{I}$. Since $\hat{\sigma}(i) = \sigma(i)$, good x must be available at (σ, ψ) in $\mathscr{E}$. As $x \notin X_0 \cup \{x_0\}$, $x \in X_f$ for some $f \in F_k \subseteq F$ and $\psi(f) = W_f$. (Since ψ is efficient, any firm f producing at ψ is allocated $\psi(f) = W_f$.) There are two subcases.

If $\hat{f}_k \in \tilde{F}$, then $x \in X_{\hat{f}_k} \subseteq \hat{X}_0$. This is because x was assigned to an agent. Hence, it must either have infinite capacity or it must not be assigned as an input to any other firm in F_k.

Otherwise, $\hat{f}_k \in \hat{F}$. We know that $\psi(f') = W_{f'}$ for all $f' \in F_k$; otherwise, $f \in F_k$ would not be able to produce its output. Therefore, $\hat{\psi}(\hat{f}_k) = (\bigcup_{f \in F_k} W_f) \setminus (\bigcup_{f \in F_k} X_f)$ and $X_{\hat{f}_k}$ is available at $(\hat{\sigma}, \hat{\psi})$. If $x \notin X_{\hat{f}_k}$, then $q_x = 1$ and $x \in W_{f'}$ for some $f' \in F_k$. However, this implies that at (σ, ψ), $\sigma(i) = x$ and $x \in \psi(f')$—a contradiction. Thus, $x \in X_{\hat{f}_k}$.

Case 2. Suppose $x \in \hat{\psi}(\hat{F})$. Thus, there exists $\hat{f}_k \in \hat{F}$ such that $x \in \hat{\psi}(\hat{f}_k)$. Hence, $x \in \psi(f)$ for some $f \in F_k$. Since ψ is efficient, $x \in W_f$. Because (σ, ψ) is feasible, the firm producing x, say f', must produce at (σ, ψ). There are two subcases.

Suppose $f' \in F'_k$ and $\hat{f}'_k \in \tilde{F}$. Since x was assigned to firm f at ψ, x either has infinite capacity or it was not assigned as an input to any other firm in F'_k. But then, we know that $x \in X_{\hat{f}'_k} \subseteq \hat{X}_0$.

Otherwise, suppose $f' \in F'_k$ and $\hat{f}'_k \in \hat{F}$. All firms in the strongly connected component F'_k must also produce at (σ, ψ). If $F'_k = F_k$, then $x \in \hat{\psi}(\hat{f}_k)$, which is not possible. Therefore, $F'_k \neq F_k$. If all $f \in F'_k$ produce at (σ, ψ), then $\psi(f) = W_f$ for all $f \in F'_k$. Thus, $\hat{\psi}(\hat{f}'_k) = (\bigcup_{f \in F'_k} W_f) \setminus (\bigcup_{f \in F'_k} X_f)$ and $\hat{f}'_k$ produces $X_{\hat{f}'_k} = (\bigcup_{f \in F'_k} X_f) \cap \left\{x \in X \,\middle|\, |\{f \in F'_k | x \in W_f\}| < q_x\right\}$, at $(\hat{\sigma}, \hat{\psi})$. If $x \notin X_{\hat{f}'_k}$, then its capacity is one and there exists some firm $f'' \in F'_k$ such that $x \in W_{f''}$. However, above we saw that $x \in W_f$ and $f \notin F'_k$. Thus there exist two firms, f and f'', that require the same input good for production. However, this contradicts condition (B2) that was satisfied by $\mathscr{E}$.

Cases 1 and 2 imply that if $x \notin X_0 \cup \{x_0\}$, then $x \in \hat{f}_{\hat{F}}(\hat{\psi})$. Hence, $\hat{\sigma}(\hat{I}) \cup \hat{\psi}(\hat{F}) \subseteq \hat{X}_0 \cup \hat{f}_{\hat{F}}(\hat{\psi}) \cup \{x_0\}$.

Next, consider good x with capacity $q_x = 1$. The following three points together imply that $|\{i \in \hat{I} | \hat{\sigma}(i) = x\}| + |\{\hat{f} \in \hat{F} | x \in \hat{\psi}(\hat{f})\}| \leq q_x$ for all $x \in \hat{X}_0 \cup \{x_0\} \cup \hat{f}_{\hat{F}}(\hat{\psi})$ and prove the claim.

(i) At most one agent can be assigned x at $(\hat{\sigma}, \hat{\psi})$. This is because (σ, ψ) is feasible and $\sigma = \hat{\sigma}$.

(ii) At most one firm can be assigned x at $(\hat{\sigma}, \hat{\psi})$. To see this, suppose $\hat{f}_k$ and $\hat{f}_\ell$ are both assigned x at $(\hat{\sigma}, \hat{\psi})$. This implies there exist two distinct firms $f_k \in F_k$ and $f_\ell \in F_\ell$ such that $x \in \psi(f_k)$ and $x \in \psi(f_\ell)$. But this means that both firms would not be able to produce at $\overline{\gamma}$, contradicting (B2).

(iii) It is impossible for an agent and a firm to be simultaneously assigned x at $(\hat{\sigma}, \hat{\psi})$ in $\hat{\mathscr{E}}$. Suppose the contrary. If $\hat{\sigma}(i) = x$ and $x \in \hat{\psi}(\hat{f}_k)$, then there exists a firm $f \in F_k \subseteq F$ such that $x \in \psi(f)$. Thus, agent i and firm f are both assigned x at (σ, ψ), a contradiction. $\diamond$

Claim 3. *$\hat{\sigma}(i) \succ_i \hat{\mu}(i)$ for all $i \in C$.*

Proof of Claim 3. Since $\hat{\sigma} = \sigma$ and $\hat{\mu} = \mu$, it follows that $\hat{\sigma}(i) \succ_i \hat{\mu}(i)$ for all $i \in C$. $\diamond$

Claim 4. $\hat{\mu}(j) \succ_j \hat{\sigma}(j) \implies \hat{\mu}(j) \in \Omega_{\hat{\gamma}}(C|\hat{\omega}, \hat{\mu})$

Proof of Claim 4. Recall that $\Omega_\gamma(C|\omega,\mu) = \bigcup_{k=0}^{\infty} Z_k$ where $Z_0 = \omega(C)$ and $Z_k = Z_{k-1} \cup \omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})$ for all $k \geq 1$. Likewise, $\Omega_{\hat{\gamma}}(C|\hat{\omega},\hat{\mu}) = \bigcup_{k=0}^{\infty} \hat{Z}_k$ where $\hat{Z}_0 = \hat{\omega}(C)$ and $\hat{Z}_k = \hat{Z}_{k-1} \cup \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})) \cup \alpha_{\hat{\gamma}}(\hat{Z}_{k-1})$ for all $k \geq 1$. Since $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega_\gamma(C|\omega,\mu)$, $\sigma(i) = \hat{\sigma}(i) \in \hat{X} \cup \{x_0\}$ for all i, and $\mu(j) = \hat{\mu}(j) \in \hat{X} \cup \{x_0\}$ for all j, to prove the claim it suffices to show that $\Omega_\gamma(C|\omega,\mu) \cap \hat{X} \subseteq \Omega_{\hat{\gamma}}(C|\hat{\omega},\hat{\mu})$. Thus, it suffices to show that $Z_k \cap \hat{X} \subseteq \hat{Z}_k$ for all $k \geq 0$.

Let $k = 0$. If $x \in Z_0 \cap \hat{X} = \omega(C) \cap \hat{X}$, then $C^x \subseteq C$. Thus, $\hat{C}^x \subseteq C^x \subseteq C$. Which implies, $x \in \hat{\omega}(C) = \hat{Z}_0$. Proceeding by induction, suppose $Z_{k'} \cap \hat{X} \subseteq \hat{Z}_{k'}$ for all $k' \leq k-1$. Let $x \in Z_k \cap \hat{X}$. If $x \in Z_{k-1} \cap \hat{X}$, then the induction hypothesis implies that $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$. Instead, suppose $x \notin Z_{k-1}$ and $x \in (\omega(C \cup \mu^{-1}(Z_{k-1})) \cup \alpha_\gamma(Z_{k-1})) \cap \hat{X}$. There are two cases.

Case 1. Suppose $x \in \omega(C \cup \mu^{-1}(Z_{k-1}))$. Since $\mu = \hat{\mu}$, the range of $\mu(\cdot)$ is contained in $\hat{X} \cup \{x_0\}$. Thus, $\mu^{-1}(Z_{k-1}) = \mu^{-1}(Z_{k-1} \cap \hat{X}) = \hat{\mu}^{-1}(Z_{k-1} \cap \hat{X}) \subseteq \hat{\mu}^{-1}(\hat{Z}_{k-1})$. Therefore, $\omega(C \cup \mu^{-1}(Z_{k-1})) \subseteq \omega(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1}))$. And so, $C^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})$, which implies $\hat{C}^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1})$. Therefore, $x \in \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-1}))$ and $x \in \hat{Z}_k$.

Case 2. Suppose $x \in \alpha_\gamma(Z_{k-1})$. Because each firm's production function is Leontief, there exists $y^1 \in Z_{k-1}$ such that $x \in \alpha_\gamma(y^1)$. There are two subcases.

(a) $y^1 \in \hat{X}$. Thus, $y^1 \in Z_{k-1} \cap \hat{X} \subseteq \hat{Z}_{k-1}$. As y^1 is critical for x at γ, it remains so at $\hat{\gamma}$ since each production function is Leontief. Thus, $x \in \alpha_{\hat{\gamma}}(\hat{Z}_{k-1})$, which implies $x \in \hat{Z}_k$.

(b) $y^1 \notin \hat{X}$. Thus, there exists some firm $f^1 \in F_k \subseteq F$ such that $y^1 \in X_{f^1}$ and there must exist some other firm $f^0 \in F_k$ that uses y^1 as an input. Moreover, the capacity of y^1 must be one (else it would be present in $\hat{X}$). Since y^1 is a critical input for x, it follows that the firm producing x must be f^0, i.e., $x \in X_{f^0}$ and $y^1 \in W_{f^0}$

We know that $y^1 \in Z_{k-1} = Z_{k-2} \cup \omega(C \cup \mu^{-1}(Z_{k-2})) \cup \alpha_\gamma(Z_{k-2})$. If $y^1 \in Z_{k-2}$, then $x \in Z_{k-1}$, which is a contradiction. Three possibilities remain.

(i) $y^1 \in \omega(C \cup \mu^{-1}(Z_{k-2}))$. In this case, $C^y \subseteq C \cup \mu^{-1}(Z_{k-2})$. However, $f^0, f^1 \in F_k$. Thus, $y^1 \in Y_{\hat{f}_k}$. Noting (15), this implies that $\hat{C}^x \subseteq C^y$. Hence, $\hat{C}^x \subseteq C \cup \mu^{-1}(Z_{k-2})$ which (by reasoning analogous to Case 1 above) implies that $\hat{C}^x \subseteq C \cup \hat{\mu}^{-1}(\hat{Z}_{k-2})$. Therefore, $x \in \hat{\omega}(C \cup \hat{\mu}^{-1}(\hat{Z}_{k-2}))$. Thus, we can conclude that $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$.

(ii) $y^1 \in \alpha_\gamma(y^2)$ where $y^2 \in Z_{k-2} \cap \hat{X}$. This implies then $x \in \alpha_{\hat{\gamma}}(y^2)$. This is because y^2 is an input for $\hat{f}_k$ and $x \in X_{\hat{f}_k}$. However, $y^2 \in Z_{k-2} \cap \hat{X}$ implies $y^2 \in \hat{Z}_{k-2}$. Thus, $x \in \alpha_{\hat{\gamma}}(\hat{Z}_{k-2})$, which implies $x \in \hat{Z}_{k-1} \subseteq \hat{Z}_k$.

(iii) $y^1 \in \alpha_\gamma(y^2)$ where $y^2 \in Z_{k-2}$ and $y^2 \notin \hat{X}$. In this case, we can repeat the preceding argument starting at (b) either establishing that $x \in \hat{Z}_k$, as in parts (i) and (ii), or identifying a new good y^3 such that $y^3 \notin \hat{X}$, as in part (iii). As there is a finite number of goods, this argument must eventually stop and it can only stop after showing $x \in \hat{Z}_k$. $\diamond$

$\square$

Proof of Proposition 3. For each $f \in F$, let $\hat{\gamma}(f) = \gamma(f) \cap W_f$. (Recall that W_f are the necessary inputs for firm f.) The production network $\hat{\gamma}$ is efficient and ensures the same aggregate output as γ, i.e., $f_F(\hat{\gamma}) = f_F(\gamma)$. Thus, if (μ, γ) is feasible, so is $(\mu, \hat{\gamma})$. Moreover, if $x \in \alpha_\gamma(Z) \iff x \in \alpha_{\hat{\gamma}}(Z)$. This is because critical inputs necessarily belong only to W_f. Thus, $\Omega_\gamma(C|\omega, \mu) = \Omega_{\hat{\gamma}}(C|\omega, \mu)$ and the result follows. $\square$

Proof of Proposition 4. Suppose economy $\mathscr{E}$ satisfies (A1)–(A4) and (B1)–(B3) and each good has capacity one. Let $\hat{\mathscr{E}}$ be its condensation. Let $(\hat{\mu}, \hat{\gamma})$ be an outcome identified by Algorithm 1 in $\hat{\mathscr{E}}$. Suppose Algorithm 1 constructs $(\hat{\mu}, \hat{\gamma})$ in T steps. If $i \in I$ is assigned x_0 in step t, let $S^t = \{i, x_0\}$. Otherwise, let S^t be the cycle of agents and goods defining the assignment. (This is the cycle after any "trimming.") Note that Algorithm 1 in step t also identifies a set of goods $\hat{X}^t$ that are never produced. Given $(\hat{\mu}, \hat{\gamma})$, define (μ^*, γ^*) in $\mathscr{E}$ as in the proof of Theorem 3.

To define prices p^* supporting (μ^*, γ^*) as an equilibrium, we adapt David Gale's argument (Shapley and Scarf, 1974, p. 30). Let $L := |X|$ be the number of goods and assign the price $p^*_{x_0} = 0$ to the outside option. Let $\pi_1 > \cdots > \pi_T > 0$ be a set of values such that $\pi_t > L^{2L}\pi_{t+1}$. We define prices inductively. Let $P^0 = \varnothing$ and for each $t \geq 1$ let P^t be the set of goods whose prices have been defined by the end of step t. At step $t \geq 1$ define prices as follows given P^{t-1}.

Step t. If $S^t \cap X = \varnothing$, set $P^t = P^{t-1}$ and proceed to step $t+1$. Otherwise, for each $x \in S^t \cap X$ set $p^*_x = \pi_t$. Let $P^t_0 \supseteq P^{t-1}$ be the set of all goods with defined prices. Continuing inductively, at iteration $\tau \geq 1$, for each good y without a defined price yet and that satisfies $x \in X_f$ and

$y \in W_f$ for some $x \in P^t_{\tau-1}$, set $p^*_y = \pi_t/L^\tau$. Let P^t_τ be the set of all goods with defined prices. As there are L goods, there exists a $\tau^* \leq L$ such that for all $\tau \geq \tau^*$, $P^t_{\tau^*} = P^t_\tau$. Given these prices, each firm producing a good whose price has been defined maximizes profits by producing since the value of its output(s) exceeds the price it pays for all inputs.

Next consider the set of goods $\hat{X}^t$. For each $x \in \hat{X}^t$ that is produced by some firm using an input in P^t_L, set $p^*_x = \pi_t/L^{L+1}$. Now, let P^t_{L+1} be the set of goods with defined prices. Continuing inductively, at iteration $\tau \geq 1$, for each $x \in \hat{X}^t$ that is produced by some firm using an input $y \in P^t_{L+\tau-1}$ and whose price has not yet been defined, set $p^*_x = \pi_t/L^{L+\tau}$. Let $P^t_{L+\tau}$ be the set of all goods with defined prices. Continue this process until all goods in $\hat{X}^t$ have a set price. This process stops after at most L iterations. Let P^t be the set of goods with defined prices and proceed to step $t+1$. Given these prices every firm that can produce some $x \in \hat{X}^t$ maximizes profits by not producing because the price of at least one necessary input exceeds the maximum revenue the firm may earn selling its output.

The above process continues until step T. Set $p^*_x = 0$ for any x whose price remains undefined. The feasible outcome (μ^*, γ^*) is an equilibrium relative to p^*. If $x \succ_i \mu^*(i)$, then good x was removed from the market prior to i's assignment. Therefore, $p^*_x > p^*_{\mu^*(i)}$. Similarly, by construction each firm is maximizing profits. The price of its outputs exceeds its input cost. Each firm's production plan is efficient. Hence, it cannot increase profits by using fewer inputs. $\square$

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